Geometric Function Theory

Thm (ABCFGLRR) Let $X/\mathbb{C}$ a smooth curve, G a reductive group.

There exists an equivalence of categories

$$\mathbb{L}_G : D\text{-mod}(\mathrm{Bun}_G(X)) \longrightarrow \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{G}}(X))$$

which satisfies <u>certain compatibilities</u>

Our goal is to understand something about the above thm.

Let's step aside from the above therem and the mysterious objects that appear in it and let's just accept the fact that we want to study the stack $\mathrm{Bun}_G(X)$, of (fppf) G-torsors on X.

By the general pattern of algebraic geometry we may study a space $\mathfrak{X}$ by studying <u>functions</u> on it.
This study will be carried out by looking at D-modules which we should think as some kind of (categorified) functions.

1 A toy model: finite sets.

- To a finite set X we may associate associate the $\mathbb{C}$-algebra of functions $\mathbb{C}[X] = \{ f : X \to \mathbb{C} \}$ multiplication and sum are point-wise.

- Given $\pi : X \to X'$ we have <u>functorial</u> morphisms

$$\pi^* : \mathbb{C}[X'] \longrightarrow \mathbb{C}[X] \qquad \pi^*(g)(x) = g(\pi(x))$$

$$\pi_* : \mathbb{C}[X] \longrightarrow \mathbb{C}[X'] \qquad \pi_*(f)(x') = \sum_{\substack{x \in X \\ \pi(x) = x'}} f(x) \qquad \text{"integrate along fibers"}$$

- This machinery allows us to perform the following:

def. A correspondence (or a span) between X and X' is a diagram

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_{X'}"] & \\ X & & X' \end{tikzcd}$$

rmk Any correspondence induces "an action of $\mathbb{C}[Z]$" between $\mathbb{C}[X]$ and $\mathbb{C}[X']$. That is a linear map (integral transform)

$$\mathbb{C}[Z] \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[X'])$$
$$K \longmapsto \left(f \longmapsto K * f = (\pi_{X'})_*(K \cdot \pi_X^* f) \right)$$

"integral kernel"

This pattern is analogous to the Fourier transform

ex. When $X = X'$ and $Z = X \times X$ the universal corr. we obtain an isomorphism

$$\mathbb{C}[X \times X] \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X]) \qquad \text{"Matrices = linear endomorphisms"}$$

Turns out we can characterize the algebra structure on the RHS via push and pulls. This is known as the <u>convolution product</u>. We state it more precisely in the relative case.

Prop. Let $\pi : X \to Y$ be a map of finite sets. $\mathbb{C}[X]$ is naturally a $\mathbb{C}[Y]$-algebra via π^*. Then

1) The integral transform $\mathbb{C}[X \times_Y X] \longrightarrow \mathrm{End}_{\mathbb{C}[Y]}(\mathbb{C}[X])$ is an iso in $\mathrm{Vect}_{\mathbb{C}}$.

2) The algebra structure on the right may be recovered from the following convolution product:

$$\begin{tikzcd} & X \times_Y X \times_Y X \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] \arrow[r, "\pi_{13}"] & X \times_Y X \\ X \times_Y X & & X \times_Y X \end{tikzcd}$$

$$V * W = (\pi_{13})_*(\pi_{12}^* W \cdot \pi_{23}^* V) \underset{\text{Claim}}{=} V \circ W$$

$$\cap$$

$$\mathbb{C}[X \times_Y X]$$

proof We may write $X = \bigsqcup_{y \in \operatorname{Im}\pi} X_y$ as the disj. union of the fibers so that $X \times_Y X = \bigsqcup_{y \in \operatorname{Im}\pi} X_y \times X_y$.

Then $\operatorname{End}_{\mathbb{C}[Y]}(\mathbb{C}[X]) = \prod_{y \in \operatorname{Im}\pi} \operatorname{End}_{\mathbb{C}}(\mathbb{C}[X_y])$ and $\mathbb{C}[X \times_Y X] = \prod_{y \in \operatorname{Im}\pi} \mathbb{C}[X_y \times X_y]$ can be identified with the algebra of block diagonal matrices, indexed by $\operatorname{Im}\pi$. (1) follows immediately.

We now prove point (2). We need to establish that

$$V * (W * M) = (V * W) * M$$

Integral action

Convolution product in the statement.

Recall that the action of $\mathbb{C}[X \times_Y X]$ on $\mathbb{C}[X]$ is given by

$$V * M = (\pi_2)_*(V \cdot \pi_1^* M),$$

Thus we need to check that

$$(\pi_2)_*\Big(G \cdot \pi_1^*\big((\pi_2)_*(F \cdot \pi_1^* h)\big)\Big) = (\pi_2)_*\Big((\pi_{13})_*\big(\pi_{12}^* F \cdot \pi_{23}^* G\big) \cdot \pi_1^* h\Big)$$

Lemma / exercise The following hold:

1) (Projection formula) For any $\pi : X \to Y$, $f \in \mathbb{C}[X]$, $g \in \mathbb{C}[Y]$

$$(\pi_* f) \cdot g = \pi_*(f \cdot \pi^* g)$$

2) (Base change formula) For any cartesian diagram of finite sets

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

the linear maps $\mathbb{C}[X] \to \mathbb{C}[Y']$

$$(g_Y)^*(f_*) = (f')_*(g_X)^*$$ coincide

Iterations of these formulas will provide the proof, considering the cartesian diagram

$$\begin{tikzcd} X\times_Y X\times_Y X \arrow[r, "\pi_{23}"] \arrow[d, "\pi_{12}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X\times_Y X \arrow[d, "\pi_1"] \\ X\times_Y X \arrow[r, "\pi_2"'] & X \end{tikzcd}$$

functoriality

$$(\pi_2)_\times\left((\pi_{13})_\times(\pi_{12}^\times W\cdot\pi_{23}^\times V)\cdot\pi_1^\times M\right) = (\pi_2)_\times(\pi_{13})_\times\left((\pi_{12}^\times W\cdot\pi_{23}^\times V)\cdot\pi_{13}^\times\pi_1^\times M\right)$$

$$= (\pi_2)_\times(\pi_{23})_\times\left((\pi_{12}^\times W\cdot\pi_{23}^\times V)\cdot\pi_{12}^\times\pi_1^\times M\right)$$

π^* of algebras

$$= (\pi_2)_\times(\pi_{23})_\times\left(\pi_{12}^\times(W\cdot\pi_1^\times M)\cdot\pi_{23}^\times V\right)$$

$$= (\pi_2)_\times\left((\pi_{23})_\times(\pi_{12})^\times(W\cdot\pi_1^\times M)\cdot V\right)$$

$$= (\pi_2)_\times\left((\pi_1)^\times\left((\pi_2)_\times(W\cdot\pi_1^\times M)\right)\cdot V\right)$$

□

Independently from the previous proposition one can show by the same means that the convolution product is associative.

Prop On $\mathbb{C}[X\times_Y X]$ the convolution product is associative.

Proof. We need to show $F\star(G\star H)=(F\star G)\star H$, that is

$$(\pi_{13})_\times\left(\pi_{12}^\times F\cdot\pi_{23}^\times\left((\pi_{13})_\times(\pi_{12}^\times G\cdot\pi_{23}^\times H)\right)\right) = (\pi_{13})_\times\left(\pi_{12}^\times\left((\pi_{13})_\times(\pi_{12}^\times F\cdot\pi_{23}^\times G)\right)\cdot\pi_{23}^\times H\right)$$

Consider the cartesian square

$$\begin{tikzcd} (a,b,c,d) \arrow[rr, mapsto] & & (b,c,d) & \\ X\times_Y X\times_Y X\times_Y X \arrow[r, "\pi_{234}"] \arrow[d, "\pi_{124}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X\times_Y X\times_Y X \arrow[d, "\pi_{13}"] & & (a,b,c) \arrow[d, mapsto] \\ X\times_Y X\times_Y X \arrow[r, "\pi_{23}"'] & X\times_Y X & & (a,c) \\ (a,b,c) \arrow[r, mapsto] & (b,c) & & \end{tikzcd}$$

(a,b,d)

$$(\pi_{13})_*\left(\pi_{12}^*F\cdot\pi_{23}^*\left((\pi_{13})_*\left(\pi_{12}^*G\cdot\pi_{23}^*H\right)\right)\right)$$

$$= (\pi_{13})_*\left(\pi_{12}^*F\cdot(\pi_{124})_*\left((\pi_{234})^*\left(\pi_{12}^*G\cdot\pi_{23}^*H\right)\right)\right)$$

$$= (\pi_{13})_*\left(\pi_{12}^*F\cdot(\pi_{124})_*\left(\pi_{23}^*G\cdot\pi_{34}^*H\right)\right)$$

$$= (\pi_{13})_*(\pi_{124})_*\left(\pi_{124}^*\pi_{12}^*F\cdot\pi_{23}^*G\cdot\pi_{34}^*H\right)$$

$$= (\pi_{14})_*\left(\pi_{12}^*F\cdot\pi_{23}^*G\cdot\pi_{34}^*H\right)$$ and the same with the RHS $\square$

rmk We notice that for these property to hold it is enough to have the following. Some category of "spaces" S, which has fibered products (and a terminal object).

0.) An assignment $X \rightsquigarrow F(X)$ where $F(X)$ is some kind of mathematical object which we may think as "functions on X".

1) $F(X)$ should be some sort of "commutative algebra object" so that it has a notion of sum $\oplus$ and product $\otimes$. Which satisfy the usual commutativity, associativity and distributive laws.

2) For any $\pi: X \to Y$ we want to have functorial (in π) morphisms

$\pi^*: F(Y) \to F(X)$ which should be a map of algebras

$\pi_*: F(X) \to F(Y)$ which should be linear

PF) An "equality" $(\pi_* f)\otimes g \simeq \pi_*(f\otimes\pi^*g)$
(Projection formula)

BCF) Given a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

an "equality" $(f')_*(g_X)^* \simeq (g_Y)^*(f)_*$

(Base Change formula)

def. This is called in the literature a "function theory".

$1\frac{1}{2}$ Geometric Actions

Fix a category of spaces S with a function theory F.

ex. Correspondence actions

Consider $\pi : X \to Y$ and a correspondence

$$\begin{tikzcd} & X \times_Y X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z & & Z \end{tikzcd}$$

Such that $p_1 \circ \pi_{13} = p_1 \circ \pi_{12}$
$p_2 \circ \pi_{23} = p_2 \circ \pi_{13}$

and such that the diagram

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{23}"'] & X \times_Y X \arrow[d, "p_2"] \\ X \times_Y X \arrow[r, "p_1"'] & Z \end{tikzcd}$$

is cartesian. (i.e. a fib. product square)

Then the formula
$$\underset{F(X \times_Y X)}{\underset{\cap}{V}} \star \underset{F(Z)}{\underset{\cap}{M}} = (p_2)_*\left(V \otimes p_1^* M\right)$$

defines an action of $F(X \times_Y X)$ on $F(Z)$, that is:

$$\text{(a)} \quad V \star (W \star M) = (V \star W) \star M \quad \text{(b)}$$

pf We need to compare the expressions

(a) $(p_2)_*\left(V \otimes p_1^*(p_2)_*\left(W \otimes p_1^* M\right)\right) = (p_2)_*\left(V \otimes (\pi_{23})_*(\pi_{12}^*)\left(W \otimes p_1^* M\right)\right)$

$= (p_2)_*(\pi_{23})_*\left(\pi_{23}^* V \otimes \pi_{12}^* W \otimes \pi_{12}^* p_1^* M\right)$

$= (p_2 \circ \pi_{23})_*\left(\pi_{23}^* V \otimes \pi_{12}^* W \otimes (p_1 \circ \pi_{12})^* M\right)$

(b) $(p_2)_*\left((\pi_{13})_*\left(\pi_{12}^* W \otimes \pi_{23}^* V\right) \otimes p_1^* M\right) = (p_2)_*(\pi_{13})_*\left(\pi_{23}^* V \otimes \pi_{12}^* W \otimes \pi_{13}^* p_1^* M\right)$

$= (p_2 \circ \pi_{13})_*\left(\pi_{23}^* V \otimes \pi_{12}^* W \otimes (p_1 \circ \pi_{13})^* M\right)$

Index: base change | projection | functoriality | π^* of algs

ex **Action on fibers** Consider $\pi : X \to Y$ and $y : pt \to Y$

Let us denote by X_y the pullback

$$\begin{tikzcd} X_y \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\pi"] \\ pt \arrow[r, "y"'] & Y \end{tikzcd}$$

Assume that

1.1) Pullbacks are stable under composition, that is

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd} \Longrightarrow \begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ Y'' \arrow[r] & Y \end{tikzcd}$$

1.2) Pullbacks are distributive : $(X_1 \times_Y X_2) \times Y' = (X_1 \times_Y Y') \times_{Y'} (X_2 \times_Y Y')$

2) $\iota : X_y \to X$ satisfies $\iota^* \iota_* \iota^* = \iota^* : F(X) \to F(X_y)$

Then the formula
$$F(X \times_Y X) \times F(X_y) \longrightarrow F(X_y)$$
$$V, M \longmapsto V \underset{y}{\star} M = \iota^* (\pi_2)_* \left(V \otimes \pi_1^* \iota_* M \right)$$

defines an action of $F(X \times_Y X)$ on $F(X_y)$.

pf We need to compare $\underset{(a)}{V \underset{y}{\star} (W \underset{y}{\star} M)} = \underset{(b)}{(V \star W) \underset{y}{\star} M}$

(b) $\iota^* (\pi_2)_* \Big((\pi_{13})_* \left(\pi_{12}^* W \otimes \pi_{23}^* V \right) \otimes \pi_1^* \iota_* M \Big) =$

$= \iota^* (\pi_2)_* (\pi_{13})_* \left(\pi_{12}^* W \otimes \pi_{23}^* V \otimes \pi_{13}^* \pi_1^* \iota_* M \right)$

$= \iota^* (\pi_3)_* \left(\pi_{12}^* W \otimes \pi_{23}^* V \otimes \pi_1^* \iota_* M \right)$

$$\text{(a)}\quad \iota^*(\pi_2)_*\left(V\otimes \pi_1^*\iota_*\iota^*(\pi_2)_*\left(W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*\left(V\otimes (\iota^2)_*(\pi_1^y)^*\iota^*(\pi_2)_*\left(W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*\left(V\otimes (\iota^2)_*(\iota^2)^*(\pi_1)^*(\pi_2)_*\left(W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*\left(V\otimes (\iota^2)_*(\iota^2)^*(\pi_{23})_*(\pi_{12})^*\left(W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*\left(V\otimes (\iota^2)_*(\pi_{23}^y)_*(\iota^3)^*\left(\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*\left(V\otimes (\pi_{23})_*(\iota^3)_*(\iota^3)^*\left(\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_2)_*(\pi_{23})_*\left(\pi_{23}^*V\otimes (\iota^3)_*(\iota^3)^*\left(\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*(\pi_3)_*(\iota^3)_*\left((\iota^3)^*\pi_{23}^*V\otimes (\iota^3)^*\left(\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)\right)$$

$$= \iota^*\iota_*(\pi_3^y)_*(\iota^3)^*\left(\pi_{23}^*V\otimes\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)$$

$$= \iota^*\iota_*\iota^*(\pi_3)_*\left(\pi_{23}^*V\otimes\pi_{12}^*W\otimes\pi_1^*\iota_*M\right)$$

$$\begin{tikzcd}
X_y^2 \arrow[r, "\iota^2"] \arrow[d, "\pi_1^y"'] & X\times_Y X \arrow[d, "\pi_1"] \\
X_y \arrow[r, "\iota"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X_y^3 \arrow[r, "\iota^3"] \arrow[d, "\pi_{23}^y"'] & X\times_Y X\times_Y X \arrow[d, "\pi_{23}"] \\
X_y^2 \arrow[r] & X\times_Y X
\end{tikzcd}$$

$$\begin{tikzcd}
X_y^3 \arrow[r, "\iota^3"] \arrow[d, "\pi_3^y"'] & X\times_Y X\times_Y X \arrow[d, "\pi_3"] \\
X_y \arrow[r, "\iota"'] & X
\end{tikzcd}$$

ex. A less naive example, always with the toy model of finite sets would be the function theory of "vector bundles".

$$\mathrm{Vect}_k(X) = \{ x \in X \mapsto V(x) \in \mathrm{Vect}_x \} \qquad \mathcal{V} \otimes \mathcal{W}(x) = \mathcal{V}(x) \otimes \mathcal{W}(x) \qquad \pi^* \mathcal{V}(x) = \mathcal{V}(\pi(x))$$

$$(\mathcal{V} \oplus \mathcal{W})(x) = \mathcal{V}(x) \oplus \mathcal{W}(x) \qquad \pi_* \mathcal{V}(y) = \bigoplus_{x \in X_y} \mathcal{V}(y).$$

We see that the equalities in the base change formula and in the projection formula become isomorphisms.

2 Hecke type algebras and the toy model of finite groupoids

We now turn to another toy model: our "spaces" will be finite groupoids. That is finite categories (finite objs and finite homs) where every morphism is an isomorphism. This should be thought as a toy model for stacks.

ex
- Every finite set is a (trivial) finite groupoid.
- Every finite set acted on by a group $G \curvearrowright X$ yields a groupoid X/G. $\mathrm{Ob}(X/G) = X$ $\quad \mathrm{Hom}(x_1, x_2) = \{ g \in G : x_2 = g x_1 \}$
- Two finite groupoids should be considered "the same" when there exists an equivalence (as categories) between them.
- Isotropy groups are relevant pt/G ($G \curvearrowright pt$ with trivial action) $\neq pt$
- A map $pt/G_1 \to pt/G_2$ is the same as a morphism of groups $G_1 \to G_2$

fact To get a good notion of fibered product (i.e. invariant under equivalence) we should consider the 2-categorical version of this.

def Fibered products are computed in the following way

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\pi"'] & Y \end{tikzcd} \qquad \mathrm{Ob}(X') = \{ (x \in X, y' \in Y', \gamma : f(x) \xrightarrow{\sim} \pi(y')) \}$$

$$\mathrm{Hom}((x_1, y_1', \gamma_1), (x_2, y_2', \gamma_2)) = \left\{ \left(\begin{matrix} x_1 \\ \downarrow \alpha \\ x_2 \end{matrix} \quad \begin{matrix} y_1' \\ \downarrow \beta \\ y_2' \end{matrix} \right) : \begin{tikzcd} f(x_1) \arrow[r, "\gamma_1"] \arrow[d, "f(\alpha)"'] & \pi(y_1') \arrow[d, "\pi(\beta)"] \\ f(x_2) \arrow[r, "\gamma_2"'] & \pi(y_2') \end{tikzcd} \right\}$$

ex/(exercise)

- The following is cartesian (i.e. a fibered product)

as a set

$$\begin{tikzcd} G \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & pt/G \end{tikzcd}$$

(follows from construction)

- Let $H \xrightarrow{\varphi} G$, then G is acted upon by $H \times H$ via $(h_1, h_2) \cdot g = \varphi(h_1) g \varphi(h_2^{-1})$. Let $H \backslash {}_{\varphi}^{G} / H$ be the associated grp.

Then the following is cartesian (i.e. a fibered product)

$$\begin{tikzcd} H \backslash {}_{\varphi}^{G} / H \arrow[r] \arrow[d] & pt/H \arrow[d, "\varphi"] \\ pt/H \arrow[r, "\varphi"'] & pt/G \end{tikzcd}$$

The map $H \backslash {}_{\varphi}^{G} / H \to pt/H$ is given by the map of sets $G \to pt$ which is equivalent with respect of the action $\circlearrowright H \times H \xrightarrow{\pi_2} H$,

rmk This describes all fibered products since every finite groupoid is equivalent to one of the form $\coprod pt/G_i$.

Let's look at functions on these spaces

def. A function on a finite groupoid X is a function $X/\sim \to \mathbb{C}$, where $X/\sim$ is the set of isomorphism classes.

rmk Still, we want to remember the groupoid structure.

ex In the case of X/G functions on this groupoid identify with functions on X which are invariant with respect to the G action.

Q While the notion of pullback is pretty clear how should the pushforward be defined? (In order to the projection and base change formulas to hold)

A Given $\varphi : H \to G$ $\quad \varphi_* : \mathbb{C}[pt/H] \to \mathbb{C}[pt/G]$ is given

$$\begin{matrix} \mathbb{C}[pt/H] & \to & \mathbb{C}[pt/G] \\ \shortparallel & & \shortparallel \\ \mathbb{C} & & \mathbb{C} \end{matrix}$$

by multiplication by $|G|/|H|$

ex. Consider
$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi"'] & pt \arrow[d, "\iota"] \\ pt \arrow[r, "\iota"'] & pt/G \end{tikzcd}$$
base change tells us
$$\iota^* \iota_* = \pi_* \pi^*$$

so that if we ask for π_* (which comes from a map of sets) to be the usual one we must have
$$\iota_*(1) = |G|$$
Considering $pt \xrightarrow{\iota} pt/G \xrightarrow{p} pt$ on the other hand, by functoriality the equation $p_* \iota_* = id_k$ gives us $p_*(1) = \frac{1}{|G|}$.

Lemma/exercise Figure out the general formulas. The projection and base change formulas hold.

example/corollary Consider finite groups $H \subset G$ then $\mathbb{C}[{}_H\backslash^{G}/_H]$ is an associative algebra. This is known as the Hecke algebra associated to the pair (G,H).

Indeed ${}_H\backslash^{G}/_H$ arises as a fibered product of the form $X \times_Y X$ (in the case $X = pt/H$, $Y = pt/G$), and the general paradigm applies.

rmk These are all very different associative algebras. For instance if $H = e$ (the trivial group) we get the group algebra $\mathbb{C}[G]$ while for $H = G$ we get the trivial algebra $\mathbb{C}$.

ex Let's compute the convolution product on $H\backslash G/H$.

$H\backslash G/H$ identifies with the following groupoid:

Choose $g_1 - g_\ell \in G$ representatives for the double coset classes.
Let $H_i = \{(h_1, h_2) \in H^2 : h_1 g_i h_2^{-1} = g_i\}$, then

$$H\backslash G/H \simeq \coprod_{i=1}^{\ell} g_i/H_i$$

Under this isomorphism the projection $\pi_j : H\backslash G/H \longrightarrow pt/H$, restricted to g_i/H_i identifies with the projection $\pi_j : H_i \longrightarrow H$ $(\longleftrightarrow pt/H_i \to pt/H)$

$$\begin{tikzcd} H_i \arrow[r] \arrow[d, phantom, "\cap"] & H \\ H^2 \arrow[ur] & \end{tikzcd}$$

Coro. The base change formula holds. Consider $H_1, H_2 \to G$.
The fibered product admits a description as before.

$$\begin{tikzcd} H_1\backslash G/H_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & pt/H_2 \arrow[d, "\varphi_2"] \\ pt/H_1 \arrow[r, "\varphi_1"'] & pt/G \end{tikzcd} \qquad \varphi_2^*(\varphi_1)_* = (\pi_2)_* \pi_1^*$$

Let $1 \in \mathbb{C}[pt/H_1]$, then $\varphi_2^*(\varphi_1)_*(1) = |G|/|H_1|$.

On the other hand write $H_1\backslash G/H_2 = \coprod g_k/H_{12,k}$ where $H_{12,k} \leq H_1 \times H_2$ is the stabilizer of g_k and $\{g_k\} \longleftrightarrow$ double coset classes.

On each $g_k/H_{12,k}$ $\pi_1^*(1) = 1$. And therefore

$$(\pi_2)_*(\pi_1^*)(1) = \sum_k (\pi_2|_{g_k/H_{12,k} \to pt/H_2})_*(1) = \sum_k |H_2|/|H_{12,k}|$$

Thus we reduced to show $|G| = \sum_k \frac{|H_1||H_2|}{|H_{12,k}|}$ but this is the orbit / stabilizer thm.

▪ Let's compute the triple fibered product

$pt/H \times_{pt/G} pt/H \times_{pt/G} pt/H$. By definition its objects are couples $(g_1, g_2) \in G^2$

And a morphism $(g_1, g_2) \to (g_1', g_2')$ is given by a triple (h_1, h_2, h_3) such that

$$\begin{tikzcd} {} \arrow[r, "g_1"] \arrow[d, "h_1"'] & {} \arrow[r, "g_2"] \arrow[d, "h_2"] & {} \arrow[d, "h_3"] \\ {} \arrow[r, "g_1'"'] & {} \arrow[r, "g_2'"'] & {} \end{tikzcd}$$

commutes.

That is
$$\begin{matrix} g_1' h_1 = h_2 g_1 \\ g_2' h_2 = h_3 g_2 \end{matrix} \iff \begin{matrix} g_1' = h_2 g_1 h_1^{-1} \\ g_2' = h_3 g_2 h_2^{-1} \end{matrix}$$

So we may identify $pt/H \times_{pt/G} pt/H \times_{pt/G} pt/H = (G \times G)/_{H \times H \times H}$

where $(h_1, h_2, h_3) \cdot (g_1, g_2) = (h_2 g_1 h_1^{-1}, h_3 g_2 h_2^{-1})$.

the two projections $(G \times G)/_{H \times H \times H} \longrightarrow H \backslash G / H$ identify with the natural ones.

Let's now turn to the case of "vector bundles".

Q What should be a vector bundle on such an object?

A It should be a <u>functor</u> $X \to \mathrm{Vect}_{\Bbbk}$, that is for any point $x \in X$ a vector space $V(x)$ and for every path $\gamma : x_1 \to x_2$ an isomorphism $\gamma_* : V(x_1) \xrightarrow{\sim} V(x_2)$ in a functorial way.

facts
- $X \rightsquigarrow \mathrm{Vect}(X)$ is an assignment $(\mathrm{Groupoids}^f)^{op} \longrightarrow \mathrm{AbCat}^{\mathrm{SymMon}}$ (abelian symm. monoidal)
- Equivalent groupoids $X_1 \underset{\varphi}{\xrightarrow{\sim}} X_2$ have equivalent vector bundle theories that is $\varphi^* : \mathrm{Vect}(X_2) \longrightarrow \mathrm{Vect}(X_1)$ is an equivalence in $\mathrm{AbCat}^{\mathrm{SymMon}}$
- Every groupoid is equivalent to one of the form $\coprod pt/G_i$ where pt/G for a finite group G is the groupoid with one object pt with $\mathrm{Aut}(pt) = G$.
- $\mathrm{Vect}(pt/G) = \mathrm{Rep}(G)$ (as abelian monoidal categories)
- $\mathrm{Hom}(pt/G_1, pt/G_2) = \mathrm{Hom}_{Gp}(G_1, G_2)$

and for $\varphi : G_1 \to G_2$

$$\begin{tikzcd} \mathrm{Vect}(pt/G_2) \arrow[r, "\varphi^*"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Vect}(pt/G_1) \arrow[d, equal] \\ \mathrm{Rep}(G_2) \arrow[r, "\mathrm{Res}_\varphi"'] & \mathrm{Rep}(G_1) \end{tikzcd}$$

Q What should the pushforward be?
(in order for the projection formula to be satisfied and base change)

ex

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & pt \arrow[d, "\iota"] \\ pt \arrow[r, "\iota"'] & pt/G \end{tikzcd}$$

$\rightsquigarrow$ $(\iota^*)(\iota_*)(k) = \pi_* \pi^* (k)$

$= \bigoplus_{g \in G} k(g)$

So we know that <u>as a vector space</u> $\iota_*(k) = \bigoplus_{g \in G} k_g$
This suggests that the correct pushforward should be $\iota_*(k) = k[G]$, the regular representation. It turns out that it should actually be its dual $\iota_*(k) = k[G]^\vee$, which, in char $= 0$ is isomorphic to the regular representation

A For $\varphi : G_1 \to G_2 \longleftrightarrow \varphi : pt/G_1 \to pt/G_2$

$$\begin{aligned} \varphi_* : \mathrm{Rep}_k(G_1) &\to \mathrm{Rep}_k(G_2) \quad \text{is the \underline{coinduction functor}} \\ V &\longmapsto \mathrm{Hom}_{k[G_1]}(k[G_2], V) \end{aligned}$$

That is the <u>right adjoint</u> to the forgetful functor.
(This is consistent with the usual algebraic geometry notation)

ex Consider $pt/G \xrightarrow{\pi} pt$. Then the associated pushforward

$\pi_* : \mathrm{Rep}(G) \to \mathrm{Vect}$ identifies with the <u>functor of invariants</u>
indeed

$$\pi_*(V) = \mathrm{coInd}_G^{pt}(V) = \mathrm{Hom}_{k[G]}(k, V) = V^G.$$

Lemma The base change formula and the projection formula hold.

pf. For the projection formula is enough to consider a single map
$\varphi : G_1 \to G_2$ (i.e. $\varphi : pt/G_1 \to pt/G_2$)

So for $V \in \mathrm{Rep}(G_1)$ and $W \in \mathrm{Rep}(G_2)$ we need to provide a (natural)
isomorphism $\varphi_*(V) \otimes W \simeq \varphi_*(V \otimes \varphi^* W)$

That is : $\mathrm{coInd}_{G_1}^{G_2}(V) \otimes W \simeq \mathrm{coInd}_{G_1}^{G_2}(V \otimes \mathrm{Res}^{G_1}_{G_2} W)$
we use adjunction, let $Z \in \mathrm{Rep}(G_2)$.

$$\begin{aligned} \mathrm{Hom}_{G_2}(Z, \mathrm{coInd}_{G_1}^{G_2}(V) \otimes W) &\simeq \mathrm{Hom}_{G_2}(Z \otimes W^\vee, \mathrm{coInd}_{G_1}^{G_2} V) \\ &\simeq \mathrm{Hom}_{G_1}(\mathrm{Res}^{G_1}_{G_2} Z \otimes (\mathrm{Res}^{G_1}_{G_2} W)^\vee, V) \\ &\simeq \mathrm{Hom}_{G_1}(\mathrm{Res}^{G_1}_{G_2} Z, V \otimes \mathrm{Res}^{G_1}_{G_2} W) \\ &\simeq \mathrm{Hom}_{G_2}(Z, \mathrm{coInd}(V \otimes \mathrm{Res}^{G_1}_{G_2} W)) \end{aligned}$$

For the base change formula it is enough to consider a diagram

$$\begin{tikzcd} H_1\backslash G/H_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & pt/H_2 \arrow[d, "\varphi_2"] \\ pt/H_1 \arrow[r, "\varphi_1"'] & pt/G \end{tikzcd}$$

Take $V \in Rep(H_1)$

- $\varphi_2^*(\varphi_1)_* V = Res^{H_2}_{G}\, coInd^{G}_{H_1} V = Hom_{H_1}(k[G], V)$
- Write $H_1\backslash G/H_2 = \coprod g_k/H_{12,k}$

then $(\pi_1^* V)|_{g_k/H_{12,k}} = Res^{H_{12}}_{H_1} V$ and

$$(\pi_2)_*(\pi_1^* V) = \bigoplus_k coInd^{H_2}_{H_{12,k}} Res^{H_{12}}_{H_1} V$$
$$= \bigoplus_k Hom_{H_{12,k}}(k[H_2], V)$$

Let us note that there exists a canonical map

$$Res^{H_2}_{G}\, coInd^{G}_{H_1} V \longrightarrow \bigoplus_k coInd^{H_2}_{H_{12,k}} Res^{H_{12,k}}_{H_1} V$$

obtaind by adjunction. Recall that $f^* \dashv f_*$ so that we have unit $1 \to f_* f^*$ and counit $f^* f_* \to 1$.

From the equality $\pi_1^*\varphi_1^* = \pi_2^*\varphi_2^*$ one gets $\pi_1^*\varphi_1^*(\varphi_1)_* = \pi_2^*(\varphi_2^*)(\varphi_1)_*$

and $\pi_2^*(\varphi_2)^*(\varphi_1)_* = \pi_1^*\varphi_1^*(\varphi_1)_* \xrightarrow{\pi_1^*(\text{counit})} \pi_1^*$ and again by adjunction

$$\varphi_2^*(\varphi_1)_* \longrightarrow (\pi_2)_*\pi_1^*$$

This being an isomorphism is a consequence of Mackey's formula. We leave it as an exercise.

rmk

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & pt \arrow[d] \\ pt \arrow[r] & pt/G \end{tikzcd}$$

is cartesian where G is considered just as a set. The triple product

$$pt \times_{pt/G} pt \times_{pt/G} pt \simeq G \times G$$

and the map $G \times G \xrightarrow[\pi_{13}]{} G$ coincides with multiplication.

⇝ $Vect(G)$ is naturally a monoidal category where
$$V \otimes W\,(g) = \bigoplus_{xy=g} V(x) \otimes V(y).$$

ex (Hecke categories) Let $H \subset G$ and consider

$$\begin{tikzcd} H\backslash G/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & pt/H \arrow[d] \\ pt/H \arrow[r] & pt/G \end{tikzcd}$$

Where $H\backslash G/H$ is the groupoid obtained by considering G with its $H \times H$ action.

Then $Vect_k(H\backslash G/H)$ is a monoidal category where the product is the "usual" convolution product.

! A significant change happens at this higher categorical level. It is a theorem of Müger and Ostrik that these categories are all Morita equivalent.

That means that when looking at "modules" for these (categorified) algebras we get, for any $H, K \subseteq G$,

$$Vect(H\backslash G/H)\text{-mod} \simeq Vect(K\backslash G/K)\text{-mod}$$

! Of course one should define what is a module in this context.

In the extreme case where $H = pt$, $H = G$ we get

$$Vect_k(G)\text{-mod} = Rep(G)\text{-mod}.$$