LOCAL SYSTEMS & (D-MODULES)

0. A bit of history

We are going to talk about holomorphic differential equations.

Rmk Let $\mathbb{D}$ be an holomorphic disk and consider a homogeneous linear system of differential equations

$$\begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad df = A \cdot f \qquad A \in \Gamma(\mathbb{D}, \Omega^1_{\mathbb{D}})$$

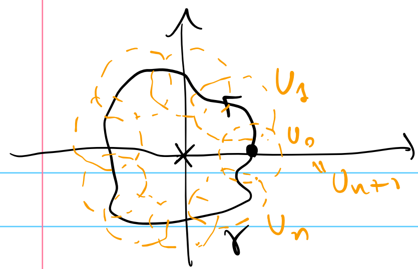
Then given $(\lambda_1 - \lambda_n) \in \mathbb{C}^n$ $\exists!$ solution such that $f_i(0) = \lambda_i$.
This may be checked using Taylor expansions.

Now consider the differential equation

$$f'(z) = \frac{1}{2z} f(z) \qquad \text{on} \qquad \mathbb{C}^\times$$

One may check that $(f^2)'' = 0$ so $f^2 = a(z-b)$ so that f is a square root function.

It has no global solution. Nevertheless we can solve it locally on small disks in a unique way.
Choose a number a_0 and a circle $\gamma : [0,1] \to \mathbb{C}^\times$



Cover γ by a finite number of disks and solve the eq. on U_0 by imposing $f(1) = a_0$.

After a loop we may end up with another value for $f \in \mathcal{O}(U_{n+1})$, $f(1) = ?$

fact ① This construction induces an action $\pi_1(\mathbb{C}^\times, 1) \longrightarrow GL(\mathbb{C}) = \mathbb{G}_m$.

② The same reasoning can be carried out for arbitrary domains $X \subseteq \mathbb{C}^n$.

To a dim $= n$ system of differential equations we can construct a <u>monodromy action</u>

$$\underset{\substack{\cap \\ M(n, \Omega^1_X)}}{df = Af} \quad \rightsquigarrow \quad \rho_{A,X} : \pi_1(X, x_0) \longrightarrow GL_n(\mathbb{C})$$

Q. (Hilbert, 1900, 21st problem) Given any $\rho : \pi_1(X, x_0) \longrightarrow GL_n(\mathbb{C})$

can we find a system of differential equations with that monodromy?

A If we allow a more general notion of differential equation the answer is yes.

§1 Connections.

Let X be a complex (holomorphic) manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on it and Ω^1_X the sheaf of holomorphic 1-forms.

We identify vector bundles on X with locally free sheaves of $\mathcal{O}_X$-modules.

def. Let $\mathcal{V}$ be a vector bundle on X.
A <u>connection</u> on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X$$

such that for any $f \in \mathcal{O}_X$ $\sigma \in \mathcal{V}$ the Leibnitz identity holds:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df.$$

A local section $\sigma \in \mathcal{V}$ is called <u>flat</u> if $\nabla(\sigma)$

local picture If $\mathcal{V} = \mathcal{O}_X^n$ then $\mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X \simeq (\Omega^1_X)^n$

and ∇ is determined by $\nabla(e_i)$ by the Leibnitz rule and one can prove that

$$\nabla = d - A \qquad A \in M(n, \Omega^1_X(X))$$

rmks ∇ induces a morphism

$$\mathrm{act}_\nabla : T_X \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{V}) \text{ via the pairing}$$

$$T_X \otimes_{\mathcal{O}_X} \mathcal{V} \xrightarrow{1\!\!1 \otimes \nabla} T_X \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V} \xrightarrow{\langle\,\rangle \otimes 1\!\!1} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{V}$$

where $\langle -,- \rangle : T_X \otimes_{\mathcal{O}_X} \Omega^1_X \longrightarrow \mathcal{O}_X$ is the canonical pairing.

This is $\mathcal{O}_X$-linear and maps into $\mathrm{Der}_{\mathcal{O}_X/\mathbb{C}}(\mathcal{V})$ the sheaf of derivations

We consider $\mathcal{O}_X$ acting on the target of $\mathrm{End}_{\mathbb{C}}(\mathcal{V})$

def • The curvature of a connection is the composition

$$\nabla^2 = \nabla \circ \nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \mathcal{V} \otimes \Omega^2_X \quad \text{where}$$

$\nabla : \mathcal{V} \otimes \Omega^1_X \longrightarrow \mathcal{V} \otimes \Omega^2_X$ is $\nabla(\sigma \otimes \omega) = \sigma \otimes d\omega - \nabla(\sigma) \wedge \omega$

• a connection is called flat if $\nabla^2 = 0$

prop. A connection is flat iff $T_X \xrightarrow{act_\nabla} \mathrm{End}_{\mathbb{C}}(\mathcal{V})$ is a Lie algebra homomorphism.

rmk This is what is called a $\mathcal{D}_X$-module on X.
More generally a left $\mathcal{D}_X$-module is a quasicoherent sheaf of $\mathcal{O}_X$-modules $\mathcal{V}$ equipped with an $\mathcal{O}_X$-linear map

$T_X \longrightarrow \mathrm{Der}_{\mathcal{O}_X/\mathbb{C}}(\mathcal{V})$ which is a morphism of Lie algs.

rmk • A connection on a vector bundle on a curve is automatically flat since $\Omega^2_X = 0$

def. We define the sheaf of solutions of $(\mathcal{V}, \nabla)$ as

$$\mathcal{V}^\nabla(U) = \{ s \in \mathcal{V}(U) \; : \; \nabla(s) = 0 \}$$

It is a sheaf of $\mathbb{C}$ vector spaces.
Locally it amounts to solve a linear homogeneous system of partial differential equations. So that the data of $(\mathcal{V}, \nabla)$ may be thought as a twisted (by $\mathcal{V}$) homogeneous system of partial differential equations.

[2] Local Systems

def A local system $\mathcal{L}$ on X is a sheaf of $\mathbb{C}$-vector spaces which is locally a constant sheaf. Constant sheaf.
$\mathcal{L}$ is said to be of finit type if $\mathcal{L} \underset{loc}{\simeq} \underline{\mathbb{C}}^n_X$ $n \in \mathbb{N}$
locally is a constant sheaf of finite dimension.

Thm [Frobenius] Let $(\mathcal{V}, \nabla)$ be a vector bundle with a flat connection. Then $\mathcal{V}^\nabla$ is a local system.

rmk In the case where X is a curve this is equivalent to the statement that on a disk $\mathbb{D}$ the solution of a diff. eq. $f' = Af$ is uniquely determined by the value of f at any point $x \in \mathbb{D}$.

Construction Let $\mathcal{L}$ be a local system of finite type then
$\mathcal{L}_X \overset{def}{=} \mathcal{L} \underset{\mathbb{C}}{\otimes} \mathcal{O}_X$ is a vector bundle and the morphism

$$\nabla_\mathcal{L} = 1\!\!1_\mathcal{L} \otimes d \;:\; \mathcal{L}_X = \mathcal{L} \underset{\mathbb{C}}{\otimes} \mathcal{O}_X \longrightarrow \mathcal{L} \underset{\mathbb{C}}{\otimes} \Omega^1_{X/\mathbb{C}} = \mathcal{L}_X \underset{\mathcal{O}_X}{\otimes} \Omega^1_{X/\mathbb{C}}$$

defines a flat connection on $\mathcal{L}_X$.

Thm (Riemann-Hilbert correspondance v1) The functors

$$\begin{array}{ccc} (\mathcal{V}, \nabla) & \longmapsto & \mathcal{V}^\nabla \\ \{\text{Vector bundles with a flat connection} / X\} & \rightleftarrows & \mathrm{LocSys}(X) \\ (\mathcal{L}_X, \nabla_\mathcal{L}) & \longleftarrow\!\!| & \mathcal{L} \end{array}$$

are inverse of each other and hence establish an equivalence.

We now compare local systems of rank n to representation of the fondamental group.

Lemma / exercise 1) Local systems on $I = [0,1]$ are trivial
2) Local systems on I^2 are trivial.

Coro Let X be a path connected simply connected space. Local systems on X are trivial.

pf sketch Let $\mathcal{L}$ be a local system on X. We want to show that $\forall x \in X$ $\mathcal{L}(X) \longrightarrow \mathcal{L}_x$ is an isomorphism
(check that this means being a trivial local system!)
For any points $x, y \in X$ we claim there is a canonical isomorphism $\varphi_{yx} : \mathcal{L}_x \longrightarrow \mathcal{L}_y$. Construct it as follows:

Lemma

- Choose a path $\gamma : x \to y$ $\gamma^*\mathcal{L}$ is trivial so there is a canonical iso $\mathcal{L}_x \simeq (\gamma^*\mathcal{L})_0 \xleftarrow{\sim} (\gamma^*\mathcal{L})([0,1]) \xrightarrow{\sim} (\gamma^*\mathcal{L})_1 = \mathcal{L}_y$
we obtain an isomorphism $\gamma_{yx} : \mathcal{L}_x \longrightarrow \mathcal{L}_y$.

- If γ' is a path homotopic to γ then $\gamma_{yx} = \gamma'_{yx}$
Indeed let $h : [0,1]^2 \to X$ such that $h(\cdot, 0) = \gamma$ $h(\cdot, 1) = \gamma'$
$h^*\mathcal{L}$ is trivial so we have the following diagram

$$\begin{tikzcd}
(h^*\mathcal{L})_{0,0} \arrow[rr, "\gamma_{yx}"] \arrow[rr, phantom, shift right=3, "?"] \arrow[dd, equal] & & (h^*\mathcal{L})_{1,0} \arrow[dd, equal] \\
& (h^*\mathcal{L})(I^2) \arrow[ul, "\sim"'] \arrow[ur, "\sim"] \arrow[dl, "\sim"] \arrow[dr, "\sim"'] & \\
(h^*\mathcal{L})_{0,1} \arrow[rr, "\gamma'_{yx}"'] & & (h^*\mathcal{L})_{1,1}
\end{tikzcd}$$

Show that the top and bottom map are actually γ_{yx}, γ'_{yx}!

- Since by assumption all paths from x to y are homotopic in X, we get the desired canonical iso $\varphi_{yx} : \mathcal{L}_x \to \mathcal{L}_y$

To prove the corollary one can show that given $s^\circ \in \mathcal{L}_x$ the collection $(s^y = \varphi_{yx} s^\circ_x)_{y \in X}$ determines a section $s \in \mathcal{L}(X)$ and that this is the only element $s \in \mathcal{L}(X)$ such that $s^\circ = s_x$.

$\square$

Thm Assume that X admits a path connected universal cover (X manifold).
The monodromy action induces an equivalence of categories

$$\mathrm{Mon}_x : \mathrm{LocSys}^{ft}(X) \longrightarrow \mathrm{Rep}^{f.d}(\pi_1(X,x))$$

Local systems of finite type

finite dimensional reps

The inverse is given by the following construction. Let $\tilde{X} \xrightarrow{\pi} X$ be the universal cover, which has an action $\pi_1(X,x) \curvearrowright \tilde{X}$ such that $\tilde{X}/\pi_1(X,x) \simeq X$.

$$\mathcal{L} : \mathrm{Rep}^{f.d.}(\pi_1(X,x)) \longrightarrow \mathrm{LocSys}^{ft}(X)$$

Quotient by the diagonal action

$$V \longmapsto \begin{cases} \mathcal{L}(V) = \tilde{X} \underset{\pi_1(X,x)}{\times} V \\ \mathcal{L}(V)(U) = \underline{V}_{\tilde{X}}(\pi^{-1}(U))^{\pi_1(X,x)} \end{cases}$$

U open $\subset X$

Constant sheaf on $\tilde{X}$

Coro (Hilbert 21) There is an equivalence of categories

$$\{\text{Vector bundles with a flat connection} / X\}$$

$$\big\downarrow \wr$$

$$\mathrm{Rep}^{f.d.}(\pi_1(X,x))$$