TANNAKIAN DUALITY

Q. How much of a group can we understand studying its category of representations?

A. Everything. This is the content of Tannakian duality.

§0 The geometric context.

We will look at functors $X : Alg_k \to Set$

ex. • $\mathbb{A}^n : Alg_k \to Set \qquad R \rightsquigarrow R^n$
• $E = \{y^2 = x^3 - 1\} : Alg_k \to Set \qquad R \rightsquigarrow \{(r_1, r_2) \in R^2 \text{ t.c. } r_2^2 = r_1^3 - 1\}$
• $\mathbb{D}_n : Alg_k \to Set \qquad R \rightsquigarrow \{r \in R : r^n = 0\}$
• $\mathbb{D} : Alg_k \to Set \qquad R \rightsquigarrow N(R)$ (nilpotent elements)

rmk The Yoneda embedding provides $Alg_k^{op} \hookrightarrow Fun(Alg_k, Set)$
& $\forall X \in Fun(Alg_k, Set)$ $\qquad A \longmapsto h_A = A.$
$Hom(h_A, F) = F(A)$

Fact. An isomorphism $h_A \xrightarrow{\sim} F$ is given by some universal element $\xi \in F(A)$

def. $X \in Fun(Alg_k, Set)$ is called affine if it's isomorphic to some h_A with $A \in Alg_k$ (or representable)

ex. $\mathbb{A}^n = h_{k[x_1, \dots, x_n]}$ $\qquad$ exercise $\mathbb{D}$ is not affine.

$$E = h_A \qquad A = \frac{k[x,y]}{y^2 = x^3 - 1}$$

$$\mathbb{D}_n = h_A \qquad A = \frac{k[t]}{t^n}$$

def. An <u>affine algebraic group scheme</u> is a functor

$G : \mathrm{Alg}_k \longrightarrow \mathrm{Grp}$ such that the underlying Set valued functor is affine.

- Equivalently one may start with an affine functor X and give a <u>functorial</u> group structure to the various $X(R)$

- Equivalently one may start with an affine functor G and give maps

$$m : G \times G \longrightarrow G \qquad (-)^{-1} : G \longrightarrow G \qquad e : pt \longrightarrow G$$

which satisfy the usual commutation.

! rmk ! As for usual groups m determines $(-)^{-1}$ and e, so that the structure of a group is equivalent to the structure of a monoid satisfying some <u>properties</u>

ex.
$\mathbb{G}_a^n : R \rightsquigarrow (R^n, +)$ $\qquad$ $\mathbb{G}_a^{\hat{e}} : R \rightsquigarrow (N(R), +)$

$\mathbb{G}_m : R \rightsquigarrow (R^\times, \cdot)$ $\qquad$ $\mathbb{G}_m^{\hat{e}} : R \rightsquigarrow (1 + N(R), \cdot)$

$GL_n : R \rightsquigarrow (GL_n(R), \cdot)$

For $V \in \mathrm{Vect}_k$ $\quad V : R \rightsquigarrow (V_R, +)$ where $V_R = V \otimes_k R$

$GL(V) : R \rightsquigarrow (GL_R(V_R), \cdot)$

ex.1 Give definitions for $SL_n, O_n, \dots$

★ Give definition for PGL_n.

ex.2 Verify that $\exp : \mathbb{G}_a^{\hat{e}} \longrightarrow \mathbb{G}_m^{\hat{e}}$

(in char $= 0$) $\qquad r \longmapsto \sum_{n \geq 0} \frac{1}{n!} r^n$ is an isomorphism of group functors.

prop/ex Let G be an affine functor and let $k[G]$ be the corresponding k-algebra. Then to give the structure of a group functor to G is equivalent to give an <u>Hopf-algebra</u> structure to G.

$$m^\# : k[G] \longrightarrow k[G] \otimes_k k[G] \qquad S : k[G] \longrightarrow k[G] \qquad \varepsilon : k[G] \longrightarrow k$$

ex Write down the Hopf algebra morphisms for GL_n.

def. G aff group scheme. A representation of G is the data of $V \in \mathrm{Vect}_k$ together with a group homomorphism

$\rho : G \to GL(V)$ ⇝ Category $\mathrm{Rep}(G)$ **ex.** Show it is abelian

prop/ex To give a rep (V, ρ) is equivalent to give a $k[G]$-comodule structure to V.

$$\rho^* : V \longrightarrow V \otimes_k k[G]$$

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowright"] & V\otimes k[G] \arrow[d, "1\otimes m^*"] \\ V\otimes k[G] \arrow[r, "\rho\otimes 1"'] & V\otimes k[G]\otimes k[G] \end{tikzcd} \quad + \quad \begin{tikzcd} V \arrow[d] \arrow[dd, bend right, equal] \\ V\otimes k[G] \arrow[d, "1\otimes e"] \\ V \end{tikzcd}$$

Prop. Every representation of an affine group scheme is the union of its finite dimensional sub-representation.

def. $\mathrm{rep}(G)$ the category of finite dimensional G-representation

---

§1 MONOIDAL CATEGORIES

def. A symmetric monoidal category is a category $\mathcal{C}$ with a functor

$\_\otimes\_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative and commutative up to natural transformations

So part of the data are natural transformations

$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$

$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X \qquad \sigma^2 = \mathrm{id}_{X \otimes Y}$

Such that the following diagrams commute

$$\begin{tikzcd}
X\otimes(Y\otimes(Z\otimes W)) \arrow[rr, "id_X\otimes\alpha_{Y,Z,W}", "\sim"'] \arrow[d, "\alpha_{X,Y,Z\otimes W}", "\wr"'] & & X\otimes((Y\otimes Z)\otimes W) \arrow[d, "\alpha_{X,Y\otimes Z,W}", "\wr"'] \\
(X\otimes Y)\otimes(Z\otimes W) \arrow[dr, "\alpha_{X\otimes Y,Z,W}"'] & \circlearrowright & (X\otimes(Y\otimes Z))\otimes W \arrow[dl, "\alpha_{X,Y,Z}\otimes id_W", "\sim"'] \\
& ((X\otimes Y)\otimes Z)\otimes W &
\end{tikzcd}$$

Pentagon identity

Hexagon identity

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[rr, "1_X\otimes\sigma_{Y,Z}"] \arrow[d, "\alpha_{X,Y,Z}"] & & X\otimes(Z\otimes Y) \arrow[d, "\alpha_{X,Z,Y}"] \\
(X\otimes Y)\otimes Z \arrow[d, "\sigma_{X\otimes Y,Z}"'] & & (X\otimes Z)\otimes Y \arrow[d, "\sigma_{X,Z}\otimes id_Y"] \\
Z\otimes(X\otimes Y) \arrow[rr] & & (Z\otimes X)\otimes Y
\end{tikzcd}$$

- We also require the existence of a <u>unity object</u> $U \in \mathcal{C}$.
That is, an object $U \in \mathcal{C}$ with an isomorphism $u : U \to U \otimes U$
such that $X \rightsquigarrow U\otimes X$ is an equivalence.
It follows that there exist natural isomorphisms $U\otimes X \simeq X \simeq X \otimes U$
which are compatible with u, α and σ.
The unit (U,u) is determined uniquely up to natural isomorphism.

Not.! We write 1 instead of U, most of the times.

def A morphism of symmetric monoidal categories $(\mathcal{C},\otimes) \to (\mathcal{D},\otimes')$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural isomorphism

$$F(X\otimes Y) \xrightarrow{\sim} F(X)\otimes F(Y)$$

compatible with α and σ.

examples. (fdim.)
- Vect_k, vect_k
- $\mathrm{Rep}_k(G)$, $\mathrm{rep}_k(G)$
- A-mod, Proj(A) (A noeth), Flat(A)
- $\mathrm{Bun}(X) = \{$vector bundles on a variety$\}$
- $\{$coverings of a topological space$\}$ $F \otimes G = F \times_X G$.

rmk In some of these categories there are two important notions

1. An internal Hom ($X, Y \in \mathrm{Vect}_k$ $\quad \mathrm{Hom}(X,Y) \in \mathrm{Vect}_k$, same for $\mathrm{rep}_k(G)$)
2. Dual objects (in vect_k (or $\mathrm{rep}_k(G)$))

def. Let $\mathcal{C}$ be a (symmetric) monoidal category. If the functor

$$T \mapsto \mathrm{Hom}(T \otimes X, Y) \qquad \mathcal{C}^{op} \to \mathrm{Set}$$

is representable we denote $\underline{\mathrm{Hom}}(X,Y)$ a representing object, any natural isomorphism (by Yoneda) $\mathrm{Hom}(T, \underline{\mathrm{Hom}}(X,Y)) \simeq \mathrm{Hom}(T \otimes X, Y)$ comes equipped with an "evaluation map" $ev_{X,Y} : \underline{\mathrm{Hom}}(X,Y) \otimes X \to Y$ which is universal.

examples. Vect_k, A-mod, Bun(X)

rmk If we assume the existence of $\underline{\mathrm{Hom}}(X,Y)$ $\forall X, Y \in \mathcal{C}$ then there is a natural "composition map"

$$\underline{\mathrm{Hom}}(Y,Z) \otimes \underline{\mathrm{Hom}}(X,Y) \longrightarrow \underline{\mathrm{Hom}}(X,Z)$$

Induced by $\underline{\mathrm{Hom}}(Y,Z) \otimes \underline{\mathrm{Hom}}(X,Y) \otimes X \xrightarrow{1 \otimes ev_{X,Y}} \underline{\mathrm{Hom}}(Y,Z) \otimes Y \xrightarrow{ev_{Y,Z}} Z$

rmk $\mathrm{Hom}(1, \underline{\mathrm{Hom}}(X,Y)) = \mathrm{Hom}(1 \otimes X, Y) = \mathrm{Hom}(X,Y)$

def The dual $X^\vee$, of an object $X \in \mathcal{C}$, is defined to be $\underline{Hom}(X, 1)$ it comes with a natural pairing $X \otimes X^\vee \xrightarrow[ev_X]{} 1$, inducing an isomorphism

$$Hom(T, X^\vee) = Hom(T \otimes X, 1)$$

rmk One can make the association $X \rightsquigarrow X^\vee$ to be a contravariant functor.

Indeed for any $f : X \to Y$ there exists a unique ${}^t f : Y^\vee \to X^\vee$ such that

$$\begin{tikzcd} Y^\vee \otimes X \arrow[r, "1\otimes f"] \arrow[d, "{}^t f \otimes 1"'] & Y^\vee \otimes Y \arrow[d] \\ X^\vee \otimes X \arrow[r] & 1 \end{tikzcd} \qquad \langle {}^t f(y), x \rangle_X = \langle y, f(x) \rangle_Y$$

rmk There is a canonical morphism $\iota_X : X \to X^{\vee\vee}$ induced by

$$Hom(X, \underline{Hom}(X^\vee, 1)) = Hom(X \otimes X^\vee, 1) \ni ev_X$$

def An object $X \in \mathcal{C}$ is reflexive if $\iota_X : X \to X^{\vee\vee}$ is an isomorphism.

ex./def An object $L \in \mathcal{C}$ is said to be invertible if $\exists\, L^{-1} \in \mathcal{C}$ together with an isomorphism $L \otimes L^{-1} \xrightarrow{\sim} 1$

(ex) $\rightarrow$ every invertible object is also reflexive and $L^{-1} \simeq L^\vee$.

examples
- $vect_k$ is the subcategory of reflexive objects of $Vect_k$
- reflexive objects in A-mod are projective ones, thus not an abelian category anymore
- $rep_k G$ is the subcategory of reflexive objects of $Rep_k G$

def. An object $X \in \mathcal{C}$ is dualizable if $\exists X^\vee \in \mathcal{C}$ toghether with morphisms

$$\epsilon : X^\vee \otimes X \longrightarrow 1$$
$$\eta : 1 \longrightarrow X \otimes X^\vee$$

such that
$$X \xrightarrow{\eta \otimes 1} X \otimes X^\vee \otimes X \xrightarrow{1 \otimes \epsilon} X = id_X$$
$$X^\vee \xrightarrow{1 \otimes \eta} X^\vee \otimes X \otimes X^\vee \xrightarrow{\epsilon \otimes 1} X^\vee = id_{X^\vee}$$

rmk $Hom(T, X^\vee) \longrightarrow Hom(T \otimes X, 1)$ is an isomorphism $\forall T$.

pf Write $Hom(T \otimes X, 1) \longrightarrow Hom(T, X^\vee)$

$$\alpha : T \otimes X \longrightarrow 1 \longmapsto T \xrightarrow{1 \otimes \eta} T \otimes X \otimes X^\vee \xrightarrow{\alpha \otimes 1} X^\vee$$

and verify it's the inverse $\square$

rmk By symmetry X is a dual of $X^\vee$ and is not hard to check that $\iota_X : X \to X^{\vee\vee}$ is an isomorphism

rmk Under the assumption that $(X \otimes Y)^\vee \simeq X^\vee \otimes Y^\vee$ (which holds under rigidity)
Every reflexive object is also dualizable, it is enough to define η as the dual of $\epsilon = ev_X$

$$\epsilon^\vee : 1 \longrightarrow X^{\vee\vee} \otimes X^\vee \overset{\iota_X^{-1} \otimes 1}{\simeq} X \otimes X^\vee$$

def. A symmetric monoidal category $(\mathcal{C}, \otimes)$ is called <u>rigid</u> if the following are satisfied

1) $\underline{Hom}(X, Y)$ exists for all $X, Y \in \mathcal{C}$
2) The natural morphisms

$$\underline{Hom}(X_1, Y_1) \otimes \underline{Hom}(X_2, Y_2) \longrightarrow \underline{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

are isos.

3) All objects of $\mathcal{C}$ are reflexive

<u>examples</u>. $vect_k$, $rep_k G$.

$\boxed{\S 2}$ FIBRE FUNCTORS

2.1 A long example.

Let X be a connected variety (topological) and let $Loc(X)$ be the category of (connected) coverings $E \to X$.
Consider it as a symmetric monoidal category via the operation

$$E \otimes F = E \times_X F \quad \text{(fibered product over } X)$$

Consider also Set as a symmetric monoidal category with $A \otimes B = A \times B$

<u>rmk</u> For any $x \in X$ the functor

$$\omega_x : Loc(X) \to Set \qquad E \mapsto E_x = \text{fibre of } x \text{ in } E$$

is symmetric monoidal.

<u>def</u> Let $Aut(\omega_x)$ be the group of natural automorphisms of ω_x. That is

$$Aut(\omega_x) = \{ g(E) : \omega_x(E) \xrightarrow{\sim} \omega_x(E) \text{ functorial in } E \}$$

rmk There is a canonical group homomorphism

$$\pi_1(X,x) \longrightarrow \mathrm{Aut}(\omega_x)$$

pf Given a covering $E \to X$ we get the <u>monodromy action</u> of $\pi_1(X,x)$ on E_x.

Given $\gamma \in \pi_1(X,x)$ and $e \in E_x$ $\gamma \cdot e = \tilde{\gamma}(1)$
where $\tilde{\gamma}$ is the lift of γ such that $\tilde{\gamma}(0) = e$

It is evident that given any morphism
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$
the induced map on the fibers $\varphi_x : E_x \to F_x$ is $\pi_1(X,x)$ equivariant.

$\gamma \cdot \varphi_x(e)$ is computed as $\tilde{\gamma}_F(1)$ where $\tilde{\gamma}_F$ is a lifting of γ starting at $\varphi(e)$.
$\rightsquigarrow$ $\varphi(\tilde{\gamma}_E)$ is such a lift for any $\tilde{\gamma}_E$ lift of γ starting at e.

$\square$

Prop. The map $\pi_1(X,x) \to \mathrm{Aut}(\omega_x)$ introduced above is an isomorphism.

pf By the universal property of the universal covering $\tilde{X} \to X$ any covering $E \to X$ admits a unique map

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\alpha"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

In addition it is known that the action of $\pi_1(X,x)$ on $\tilde{X}_x$ is simply transitive.

This last assertion implies that the composition
$$\pi_1(X,x) \longrightarrow \mathrm{Aut}(\omega_x) \longrightarrow \mathrm{Aut}(\omega_x(\tilde{X}))$$ is injective
so that the first map is injective.

It remains to show surjectivity, so pick an element $g \in \mathrm{Aut}(\omega_x)$, then $g(\tilde{X})$ is an automorphism of $\tilde{X}_x$.

rmk We have an identification $\pi_1(X,x) = \mathrm{Aut}(\tilde{X})$. The natural action on $\tilde{X}_x$ is simply transitive and commutes with the monodromy action.

$g : \tilde{X}_x \to \tilde{X}_x$ is $\mathrm{Aut}(\tilde{X})$-equivariant

$$\begin{tikzcd} \tilde{X}_x \arrow[r, "\gamma_x"] \arrow[d, "g"'] & \tilde{X}_x \arrow[d, "g"] \\ \tilde{X}_x \arrow[r, "\gamma_x"'] & \tilde{X}_x \end{tikzcd}$$

In particular, fix $\tilde{e}_0 \in \tilde{X}_x$ then g is determined by its value on $\tilde{e}_0$ since $g \cdot (\gamma \tilde{e}_0) = \gamma(g \cdot \tilde{e}_0)$ and $\pi_1(X,x) \curvearrowright \tilde{X}_x$ is simply transitive.

Let $\gamma_g \in \pi_1(X,x)$ such that $\gamma_g \cdot \tilde{e}_0 = g \cdot \tilde{e}_0$.
We claim $\gamma_g = g$ in $\mathrm{Aut}(\omega_x)$.

For $E \in \mathrm{Loc}(X)$ and $e \in E_x$. Let $\alpha : \tilde{X} \to E$ from the univ. property

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\alpha"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

$$g \cdot e = g \cdot (\alpha \cdot \tilde{e}) \underset{\text{funct. of } g}{=} \alpha \cdot (g \cdot \tilde{e}) \underset{\text{simply transitive}}{=} \alpha \cdot (g \cdot \underbrace{\gamma \cdot \tilde{e}_0}_{\text{Aut action}})$$

$$= \alpha \circ (\underbrace{\gamma \cdot g}_{\text{Aut action}} \cdot \tilde{e}_0) = \alpha \cdot (\underbrace{\gamma \cdot \gamma_g \tilde{e}_0}_{\text{Aut action}}) = \alpha(\gamma_g \cdot \tilde{e})$$

$$= \gamma_g \circ (\alpha \cdot \tilde{e}) = \gamma_g \cdot e.$$

$\square$

2.2 The fibre functor for $\mathrm{rep}_k G$.

def Let $\omega : \mathrm{rep}_k G \to \mathrm{vect}_k$ be the forgetful functor.
This is k-linear, exact, faithful and monoidal.

def Let $\underline{\mathrm{Aut}}^{\otimes}(\omega) : k\text{-Alg} \longrightarrow \mathrm{Grp}$ to be the functor defined by

$$\underline{\mathrm{Aut}}^{\otimes}(\omega)(R) = \mathrm{Aut}^{\otimes}(\mathrm{rep}_k G \xrightarrow{\omega} \mathrm{vect}_k \xrightarrow{-\otimes R} \mathrm{mod}_R)$$

where $\mathrm{Aut}^{\otimes}$ of a monoidal functor $F : (\mathcal{C}, \otimes) \to (\mathcal{D}, \otimes')$ means the subset of natural isomorphisms $F \underset{\varphi}{\Rightarrow} F$ such that

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\varphi_{X \otimes Y}} & F(X \otimes Y) \\ \downarrow & \circlearrowright & \downarrow \\ F(X) \otimes F(Y) & \xrightarrow{\varphi_X \otimes \varphi_Y} & F(X) \otimes F(Y) \end{array}$$

To rephrase the definition an element $\lambda \in \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ is a collection of R-linear isomorphisms $\lambda_V : V_R \to V_R$, indexed by finite representations of G, such that, for any G-equivariant $f : V \to W$ $\begin{array}{ccc} V_R & \xrightarrow{f_R} & W_R \\ {\scriptstyle\lambda_V}\downarrow & \circlearrowright & \downarrow{\scriptstyle\lambda_W} \\ V_R & \xrightarrow{f_R} & W_R \end{array}$ commutes
$\lambda_k : R \to R$ is the identity,
and $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$

rmk. There is a canonical morphism of group functors

$$G \longrightarrow \underline{\mathrm{Aut}}^{\otimes}(\omega)$$

Thm This is an isomorphism.

pf. Let $\lambda \in \underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ then it's easy to see that the collection
$\{\lambda_V : V_R \to V_R,\ V \in \mathrm{rep}_k G\}$ uniquely extends to a family defined on the
$\{\lambda_V : V_R \to V_R,\ V \in \mathrm{Rep}_k G\}$ \ entire $\mathrm{Rep}_k G$, satisfying the same prop.

Let A be the Hopf algebra corresponding to G.
Recall that it is also a representation via
$$\Delta : A \to A \otimes A \quad (\text{previously denoted by } m^* : k[G] \to k[G] \otimes k[G])$$
recall that the comodule structure on the tensor product is given by
$$V \otimes W \xrightarrow{\rho_V \otimes \rho_W} V \otimes W \otimes A \otimes A \xrightarrow{1 \otimes 1 \otimes m} V \otimes W \otimes A$$

<u>Claim</u> $m : A \otimes A \to A$ is of representations ($r_A \otimes r_A$ vs r_A)

$$\begin{tikzcd}
A \otimes A \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "m"'] & (A \otimes A) \otimes (A \otimes A) \arrow[r, "m"] & A \otimes A \otimes A \arrow[d, "m \otimes 1"] \\
A \arrow[rr, "\Delta"'] & & A \otimes A
\end{tikzcd}$$

$$(ab, cd) \longleftarrow\!\shortmid (a,b,c,d), \; (x,z,y,z) \longleftarrow (x,y,z)$$

$$\begin{tikzcd}
G \times G & G \times G \times G \times G \arrow[l] & G \times G \times G \arrow[l] \\
G \arrow[u] & & G \times G \arrow[ll] \arrow[u]
\end{tikzcd}$$

$(x,x) \leftarrow\!\shortmid x$ $\qquad$ $(x,x,y) \leftarrow\!\shortmid (x,y)$

$$xy \longleftarrow\!\shortmid (x,y)$$

$$\begin{tikzcd}
xy, xy & x,y,x,y \arrow[l, mapsto] & x,x,y \arrow[l, mapsto] \\
xy \arrow[u, mapsto] & & x,y \arrow[ll, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

We get that $\lambda_{A \otimes A} = \lambda_A \otimes \lambda_A$:
$$\begin{tikzcd}
A_R \otimes A_R \arrow[r] \arrow[d, "m_R"] & A_R \otimes A_R \arrow[d, "m_R"] \\
A_R \arrow[r, "\lambda_A"'] & A_R
\end{tikzcd}$$
so that λ_A is an algebra isomorphism.

Next, we claim that $\Delta : A \to A \otimes A$ is a morphism btw r_A and $1 \otimes r_A$ where r_A is the regular representation

Indeed

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & A\otimes A \arrow[d, "\Delta\otimes 1"] \\ A\otimes A \arrow[r, "1\otimes\Delta"'] & A\otimes A\otimes A \end{tikzcd} \iff \begin{tikzcd} G \arrow[dr, phantom, "\circlearrowright"] & G\times G \arrow[l] \\ G\times G \arrow[u] & G\times G\times G \arrow[l] \arrow[u] \end{tikzcd} \quad \begin{matrix} (xy, z) \\ \uparrow \\ \end{matrix}$$

$$(x, yz) \longmapsfrom (x, y, z)$$

That implies that

$$\begin{tikzcd} A_R \arrow[r, "\lambda_A"] \arrow[d, "\Delta"'] & A_R \arrow[d, "\Delta"] \\ A_R\otimes A_R \arrow[r, "\substack{\lambda_1\otimes\lambda_A \\ \wr \\ \mathrm{id}\otimes\lambda_A}"'] & A_R\otimes A_R \end{tikzcd}$$

Translated in the group version, we have a map

$G_R \xrightarrow{\lambda^\vee} G_R$ (λ_A of algebras) such that

$$\begin{tikzcd} G_R\times G_R \arrow[r, "\mathrm{id},\lambda^\vee"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G_R\times G_R \arrow[d] \\ G_R \arrow[r, "\lambda^\vee"'] & G_R \end{tikzcd}$$

that means $\lambda^\vee(xy) = x\lambda^\vee(y)$

$$\begin{matrix} GL_R(A_R) \\ \cup\!\shortmid \\ \lambda_A = \lambda^\vee(1) \end{matrix}$$

And therefore $\lambda^\vee(g) = g\cdot\lambda^\vee(1)$

It follows that λ is given by

$$A_R \xrightarrow{\Delta_R} A_R\otimes A_R \xrightarrow{1\otimes\lambda^\vee(1)} A_R\otimes R = A_R$$

Now for any V let V_0 the underlying trivial rep-

$\rho: V \longrightarrow V_0\otimes A$ is of A-comodules so

$$\begin{tikzcd} V \arrow[d, "\lambda_V"'] & \subseteq V_0\otimes A \arrow[r] \arrow[d, "1\otimes\lambda_A"] & V_0\otimes A\otimes A \\ V & \subseteq V_0\otimes A & \end{tikzcd}$$

$\rightsquigarrow$ λ_V is the multiplication by $\lambda^\vee(1)$

Thm 2 Let $(\mathcal{C},\otimes)$ be a k-linear, abelian, symmetric monoidal category which is rigid. Let

$$\omega : \mathcal{C} \longrightarrow vect_k$$

be a k-linear, monoidal, exact, faithful functor.

Then $\underline{Aut}^{\otimes}(\omega) = G$ is an affine group scheme $/k$ and the canonical morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & rep_k G \arrow[d] \\ & vect_k \end{tikzcd}$$

is an equivalence.

Thm 3. The functor

$$G\text{-torsors} \longrightarrow Hom^{\otimes}(rep_k G, Bun(X)) \qquad P \longmapsto \left(V \mapsto P \times_G V = P\times V/G\right)$$

is fully faithful