TOPOLOGICAL ALGEBRAS & THEIR CATEGORIES OF MODULES

Work over a fixed field k, char $k = 0$.

[0] The abelian story

- A f.system of n. on a vector space is a cofiltered poset of subspaces $\{V_i\}_{i\in I} \subseteq V$
- $\{V_i\} \leq \{W_j\}$ if $\exists j\ \exists i(j)\ V_{i(j)} \subseteq W_j$
- $\{V_i\} \sim \{W_j\}$ if $\{V_i\} \leq \{W_j\}$ and $\{W_j\} \leq \{V_i\}$
- A topology on V is an equivalence class of fsnoz. - they are ordered as above.
- $f: V \longrightarrow W$ $\quad \tau$ topology on W $\rightsquigarrow f^*\tau = [\{f^{-1}W_i\}_{i\in I}]$ if $\tau = [\{W_i\}_{i\in I}]$
- $f: (V,\tau) \longrightarrow (W,\sigma)$ continuous iff $\tau \leq f^*\sigma$
- (V,τ) complete if for any $\tau = [\{V_i\}]$ $V \xrightarrow{\sim} \lim V/V_i$
- τ discrete if $\tau = [\{0\}]$
- V, W complete $\rightsquigarrow$ $\mathrm{Hom}^{\mathrm{cont}}(V,W) = \mathrm{Hom}^{\mathrm{cont}}(\lim V_i, \lim W_j)$

$$= \lim_j \mathrm{Hom}^{\mathrm{cont}}(\lim V_i, W_j)$$

$$= \lim_j \operatorname{colim}_i \mathrm{Hom}(V_i, W_j)$$

rmk An additive functor $F: Vect \to Vect$

def. $Pro(Vect^{\heartsuit}) = End_k(Vect^{\heartsuit})^{op}$

It can be characterized by $Fun^{cont}_k(Pro(Vect^{\heartsuit}), \mathcal{C}) \quad F \mapsto \lim_{V \in Vect/F} \varphi(V)$

$\cap$

$Fun_k(Vect^{\heartsuit}, \mathcal{C}) \quad \varphi$

rmk $Vect^{\heartsuit} \xrightarrow{h_-} End_k(Vect^{\heartsuit})^{op}$ are compact if I is a filtered system

$$Hom(\underline{\lim} F_i, h_X) = (\underline{\lim} F_i)(X) = colim_i (F_i(X)) = colim\, Hom(F_i, h_X)$$

So that if $F = \lim F_i \quad G = \lim G_j$ (representables)

$$Hom(F, G) = \lim_j colim_i Hom_{Vect}(F_i, G_j)$$

Cor The functor $Vect^{\heartsuit}_{top, compl} \longrightarrow ProVect$

$(V, \{V_i\}) \longrightarrow \lim_i V/V_i$ is fully faithful

$V \longmapsto Hom^{cont}(V, -)$

0.1 Tensor products

ass. comm $\overset{!}{\otimes}$ $\quad Vect^{\heartsuit}_{top, compl} \quad ProVect^{\heartsuit}$

$V \overset{!}{\otimes} W = \lim_{i,j} (V/V_i \otimes W/W_j) \quad Pro(\otimes) = \overset{!}{\otimes}$

ass. $\vec{\otimes}$ $\quad V \vec{\otimes} W$ is the completion of $V \otimes W$ along the topology in which open subspaces are those containing

$V \otimes W_j + \sum_{\omega \in W} V_\omega \otimes \omega \qquad V_\omega \subseteq V$ open.

0) $V \vec{\otimes} k = V$

Rmk. 1) $V \vec{\otimes} \_$ restricted to Vect preserves direct sums

$(V \vec{\otimes} (\oplus W_j)) = \oplus V \vec{\otimes} W_j$

2) $V \vec{\otimes} \_ : \text{Vect}^{\text{compl}}_{\text{top}} \to \text{Vect}^{\text{compl}}_{\text{top}}$ preserves filtered limits.

3) For $U \in \text{Vect}$ $\quad \text{Hom}^{\text{cont}}(V \vec{\otimes} W, U) = \text{Hom}^{\text{cont}}(W, \underline{\text{Hom}^{\text{cont}}(V, U)})$

discrete.

Rmk Consider $G : \text{Vect} \to \text{Vect}$. (ProVect $\ni G$)

$V \vec{\otimes} G : \text{Vect} \to \text{Vect}$ $\qquad V \vec{\otimes} G = G \circ h_V$

$U \longmapsto G(\text{Hom}^{\text{cont}}(V, U))$

If $G = h_W$ then $\quad h_{V \vec{\otimes} W} = h_W \circ h_V$

So the ProVect picture of $\vec{\otimes}$ is $F \vec{\otimes} G = G \circ F$ associative blah blah blah.

We continue this story in the setting of DG Categories.
The reason is that we will ultimately will have to make homological algebra considerations on modules over a topological algebra.

00 Motivations / Disclaimer.

(★) The ultimate interest is <u>geometric representation theory</u> so that

i) Some of the categories we are interested in will come up as categories of sheaves on schemes/prestacks. For instance we want to come up with a sheaf theory strong enough that $Sh(LG)$ is monoidal wrt convolution.

At the same time $Sh(LG)$ should remember the IndPro nature of LG

ii) Homological algebra is essential when working btw functors btw categories of sheaves so we want to mix homological algebra & topological algebra

ex Let H be a classical group functor which is an IndScheme (↑ affine) (closed immersions)

$$A = \varprojlim \Gamma(\mathcal{O}_{H_i}, H_i) = \Gamma(\mathcal{O}_H, H)$$

is a $\overset{!}{\otimes}$ algebra and hence a $\overrightarrow{\otimes}$ algebra

$m : H \times H \longrightarrow H$ induces $A \xrightarrow{m^*} A \overset{!}{\otimes} A$

Setting We will be mainly interested in the $(\infty,2)$-category of DG-Categories over a fixed ground field of characteristic 0.

We will be mostly looking at DG Categories which are accessible & cocomplete (admit arbitrary $\oplus$), $DGCat^{cont}$ denotes the category of cocomplete DG categories with (co)continuous, exact functors.

- $Vect = D(k)$, $A\text{-mod} = D(A)$ the derived DG-categories
- $DGCat^{cont}$ admits a tensor product $-\otimes-$ which makes $(DGCat^{cont}, \otimes)$ into a symmetric monoidal category. $Vect = \mathbb{1}$
- $DGCat^{cont}$ is a closed monoidal category, that is
  - There is an internal $\underline{Fun}(\mathcal{C},\mathcal{D})$ (we omit cont)
  - $Fun(\mathcal{C}\otimes\mathcal{D},\mathcal{E}) \simeq Fun(\mathcal{C}, \underline{Fun}(\mathcal{D},\mathcal{E}))$

- There exists a strict monoidal functor $QCoh : (SchAff_k, \times) \longrightarrow (DGCat^{cont}, \otimes)$ where $SchAff_k = CAlg_k^{op}$ such that

1) $QCoh(Spec\, A) \simeq A\text{-mod}$
2) $QCoh(Spec\, A \times Spec\, B) \simeq A\text{-mod} \otimes B\text{-mod} \simeq (A \otimes_k B)\text{-mod}$
3) $f : Spec\, B \to Spec\, A$

$$\begin{array}{ccc} QCoh(Spec\, A) & \xrightarrow{f^*} & QCoh(Spec\, B) \\ \wr\wr & & \wr\wr \\ A\text{-mod} & \xrightarrow{\overset{L}{\otimes}_A B} & B\text{-mod} \end{array}$$

Extendable to all prestacks. (under cpt generation)

Accessibility

def. $\mathcal{C} \in Cat$ is κ-accessible if one of the following equivalent condition hold:

1) $\mathcal{C} \simeq Ind_\kappa(\mathcal{C}_0)$ with $\mathcal{C}_0$ small

   $\subseteq Psh(\mathcal{C}_0)$ generated by $\mathcal{C}_0$ under κ-filtered colimits.

   $\mathcal{C}_0$ is κ-compact

2) $\mathcal{C}$ is
   - (i) locally small
   - (ii) Has κ-filtered colimits
   - (iii) $\mathcal{C}^\kappa \subseteq \mathcal{C}$ of κ-cpt obj is essentially small
   - (iv) $\mathcal{C}^\kappa \subseteq \mathcal{C}$ generates under κ-filtered colimits

$\kappa \leq \lambda$

λ-filtered $\Rightarrow$ κ-filtered

($D \subseteq \mathcal{C}$ $|Hom(D)| < \kappa$ $\Rightarrow$ D has a cocone.)

κ-compact $\Rightarrow$ λ-compact

def. $\mathcal{C}$ accessible $F : \mathcal{C} \to \mathcal{D}$ is κ-accessible if preserves κ-filtered colimits

def $\lambda \trianglelefteq \mu$ (cardinals) if $\forall \mu' < \mu$ $P_\lambda(\mu') = \{X \subseteq \mu' : |X| < \lambda\}$ has a cofinal subset of $|\cdot| < \mu$

$\Updownarrow$

$\{\lambda\text{-acc } Cat' \Rightarrow \mu\text{-acc. } Cat\}$

- $\lambda < \mu$ + μ inaccessible $\Rightarrow$ $\lambda \trianglelefteq \mu$
- $\lambda \leq \mu$ $\Rightarrow$ $\lambda \trianglelefteq (2^\mu)^+$
- there are arbitrarily large cardinals for which a category is

def (locally) Presentable = accessible + all small colimits

rmk Vect is presentable.

AFT $F : C \to D$ btw locally presentable categories

1) F has a right adjoint iff preserves small colimits
2) F has a left adjoint iff preserves small limits

An accessible, exact, $Vect^c$-linear functor $Vect \to Vect$ is prorepresentable.

$$F(U) = \operatorname{colim}_{i \in I} Hom(V_i, U) \qquad I \ \kappa\text{-filtered}$$

$V_i^{\kappa} \in Vect^{\kappa}$ ← generators

$$F(\operatorname{colim} U_i^{\kappa}) = \operatorname{colim} F(U_i^{\kappa})$$
$$\shortparallel$$
$$U$$

$$\operatorname{colim}_{X \in Vect^{\kappa}/F} \underline{Hom}(X,-) = F$$

$$\operatorname{colim} \underline{Hom}(X,-) \longrightarrow F \qquad X \in (Vect^{\kappa})^{op}/F$$

ok — is this κ filtered? yes since F is so

$X_1 \ \dots \ X_k \to F$

(Vect$^{\kappa}$ is closed under $\lambda < \kappa$ $\bigoplus_{\lambda}$ direct sums)

è iso? on $U = \operatorname{colim} U_i^{\kappa}$

$$\operatorname{colim}_{X} \operatorname{colim}_{I} \underline{Hom}(X, U_i^{\kappa})$$

$$\operatorname{colim}_{X \in Vect^{\kappa}/F} \underline{Hom}(X, U_i^{\kappa}) = F(U_i^{\kappa}) \quad \square$$

Prop. Let $\mathcal{C} \in DGCat^{cont}$ and $F : \mathcal{C} \to Vect$ be accessible, continuous. Then F is prorepresentable.

[1] It is possible to enhance the above discussion to Vect (DG Category)

We get $\mathrm{ProVect}, \vec{\otimes}, \overset{!}{\otimes}$. $\mathrm{ProVect} = \mathrm{End}_{\mathrm{DGCat}_{\mathrm{ls}}}(\mathrm{Vect}, \mathrm{Vect})^{op}$ ← accessible functors

fact ProVect has a natural t-structure for which
$\mathrm{Pro\,Vect}^{\leq 0} = \mathrm{Pro}(\mathrm{Vect}^{\leq 0})$ — left t-exact functors
$\mathrm{ProVect}^{\geq 0} = \mathrm{Pro}(\mathrm{Vect}^{\geq 0})$ — right t-exact functors

so $\mathrm{ProVect}^{\heartsuit} = \mathrm{Pro}(\mathrm{Vect}^{\heartsuit})$

$\mathrm{Vect}^{\leq 0} \to \mathrm{End}(\mathrm{Vect})^{op}$
$U \mapsto \mathrm{Hom}(U, -)$
↑ sends $\mathrm{Vect}^{\geq 0}$ to itself.

l. t-exact are closed under colimits ?

rmk $\mathrm{Alg}^{\vec{\otimes}} \simeq \{\text{accessible comonads on Vect}\}^{op}$

def $A\text{-mod}_{top} = A\text{-mod}(\mathrm{ProVect})$ $\quad (A \in \mathrm{Alg}^{\vec{\otimes}})$

$$A\text{-mod}_{naive} = A\text{-mod}(\mathrm{ProVect}) \times_{\mathrm{ProVect}} \mathrm{Vect} \xrightarrow{\mathrm{Oblv}} \mathrm{Vect}$$

rmk Let $\vec{A} : \mathrm{Vect} \to \mathrm{Vect}$ the comonad given by A.
Then
$$A\text{-mod}_{naive} = S\text{-comod}$$

$A\text{-mod}_{naive} \xrightarrow{\mathrm{oblv}} \mathrm{Vect}$ is continuous, conservative and comonadic.

rmk As any functor in DGCat_{cont} Oblv is pro-representable (accessible)

$\exists\, M_i : I \to A\text{-mod}_{naive}$ such that

$$\operatorname*{colim}_{i \in I} \underline{\mathrm{Hom}}_A(F_i, -) \xrightarrow{\sim} \mathrm{Oblv}$$

We claim $\lim_{i \in I} \mathrm{Oblv}(F_i) \simeq A$.

Indeed let $\mathrm{cofree}_A : \mathrm{Vect} \to A\text{-mod}_{naive}$ the right adjoint to Oblv
by the comonadic picture $\mathrm{Oblv} \circ \mathrm{cofree}_A = \underline{\mathrm{Hom}}_{\mathrm{ProVect}}(A, -)$

$$\underline{\mathrm{Hom}}_{\mathrm{ProVect}}(\lim_{i \in I} \mathrm{oblv}\, F_i, V) = \operatorname*{colim}_{I} \underline{\mathrm{Hom}}(\mathrm{oblv}\, F_i, V) =$$

$= \operatorname{colim}_I \underline{\mathrm{Hom}}(F_i, \mathrm{cofree}_A V) = \mathrm{Oblv} \circ \mathrm{cofree}_A V$

$= \underline{\mathrm{Hom}}_{\mathrm{ProVect}}(A, V)$

1.1 Tensor products

- ProVect is endowed with two monoidal structures

$\overset{!}{\otimes} = \mathrm{Pro}(\otimes)$ (commutative) $\qquad \vec{\otimes} = \overset{\mathrm{op}}{\circ}$ (under $\mathrm{ProVect} = \mathrm{End}(\mathrm{Vect})^{\mathrm{op}}$)

Let Alg(Cat) be the category of monoidal categories with lax monoidal functors (i.e. $(\mathcal{C}, \otimes_1) \xrightarrow{F} (\mathcal{D}, \otimes_2)$, $F(X) \otimes_2 F(Y) \to F(X \otimes_1 Y)$)

it is symmetric monoidal w.r.t. products.

Prop. $(\mathrm{ProVect}, \vec{\otimes}')$ is a commutative algebra in Alg(Cat) with multiplication $\overset{!}{\otimes}$

cor There is a natural transformation

$$(V_1 \overset{!}{\otimes} V_2) \vec{\otimes}' (W_1 \overset{!}{\otimes} W_2) \longrightarrow (V_1 \vec{\otimes}' W_1) \overset{!}{\otimes} (V_2 \vec{\otimes}' W_2)$$

$$(\mathrm{ProVect}, \vec{\otimes}') \times (\mathrm{ProVect}, \vec{\otimes}') \xrightarrow{\overset{!}{\otimes}} (\mathrm{ProVect}, \vec{\otimes}')$$

$$F(V, W) = V \overset{!}{\otimes} W \qquad F(V_1, V_2) \vec{\otimes}' F(W_1, W_2) = (V_1 \overset{!}{\otimes} V_2) \vec{\otimes}' (W_1 \overset{!}{\otimes} W_2)$$

$$\downarrow \qquad\qquad \downarrow$$

$$F(V_1 \vec{\otimes}' W_1, V_2 \vec{\otimes}' W_2) = (V_1 \vec{\otimes}' W_1) \overset{!}{\otimes} (V_2 \vec{\otimes}' W_2)$$

proof Day convolution

$\mathcal{D} = \mathrm{Vect}$ with the algebra structure

$$\mathrm{Hom}_{\mathrm{DGCat}}(\mathcal{C}, \mathcal{D}) \qquad \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{lax}}(\mathrm{Cat})}(\mathcal{E}, \mathrm{Hom}(\mathcal{C}, \mathcal{D})) = \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{C} \otimes \mathcal{E}, \mathcal{D})$$

($\mathcal{C} = \mathrm{Vect}$) has day convolution

Claim $\overset{!}{\otimes}$ corresponds to Day convolution

$$\mathrm{Hom}(\mathcal{C},\mathcal{D}) \times \mathrm{Hom}(\mathcal{C},\mathcal{D}) \longrightarrow \mathrm{Hom}(\mathcal{C},\mathcal{D})$$

$$\mathcal{E} \xrightarrow{(X,Y)} \mathrm{Hom}(\mathcal{C},\mathcal{D}) \times \mathrm{Hom}(\mathcal{C},\mathcal{D}), \qquad \mathcal{E} \longrightarrow \mathrm{Hom}(\mathcal{C},\mathcal{D})$$

$$\bar{X} \cdot \bar{Y} : \mathcal{C} \otimes \mathcal{E} \longrightarrow \mathcal{D}$$

~~Fa il prodotto in arrivo~~

NO

$$\mathrm{Hom}_{\mathrm{ProVect}}(V \otimes W, U) = \operatorname{colim}_{i,j} \mathrm{Hom}_{\mathrm{Vect}}(V_i \otimes W_j, U)$$

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathrm{Fun}(\mathcal{C},\mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathrm{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\bar{\otimes}} \mathrm{Fun}(\mathcal{C}\times\mathcal{C},\mathcal{D}) \xrightarrow{\mathrm{Lan}} \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

$$\mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{D}, \qquad \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}$$

pure $\mathcal{C}$ è monoidale $\quad \mathrm{Hom}_{[\mathcal{C},\mathcal{D}]}(X \otimes_{\mathrm{Day}} Y, Z) = \mathrm{Hom}_{[\mathcal{C}\times\mathcal{C},\mathcal{D}]}(X \bar{\otimes} Y, Z \circ \otimes)$

$$\mathrm{Hom}_{\mathrm{ProVect}}(V \otimes_{\mathrm{Day}} W, h_U) = \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Vect}\times\mathrm{Vect},\mathrm{Vect})}(V \bar{\otimes} W, h_U \circ \otimes)$$

$$\mathrm{LHS}(A,B) = \mathrm{Hom}^{\mathrm{cont}}(V,A) \otimes \mathrm{Hom}^{\mathrm{cont}}(W,B) = \operatorname{colim}_{i,j} \mathrm{Hom}(V_i,A) \otimes \mathrm{Hom}(W_j,B)$$

$$\mathrm{RHS}(A,B) = \mathrm{Hom}(U, A\otimes B)$$

$$\mathrm{ProVect} \subset \mathrm{End}(\mathrm{Vect})^{op}$$

Day convolution in End(Vect)

$$\mathrm{Hom}_{\mathrm{End}(\mathrm{Vect})}(V \otimes_{\mathrm{Day}} W, F) \qquad U \rightsquigarrow \mathrm{Hom}$$

- colimit preserving in each variable
- on Vect è la cosa giusta

Prop 3.3 (nCatLab)

$$\mathrm{End}(\mathrm{Vect}) \times \mathrm{End}(\mathrm{Vect}) \xrightarrow{\tilde{\otimes}} \mathrm{End}(\mathrm{Vect})$$
$$F \quad , \quad G \longmapsto (U \mapsto F(U) \otimes G(U))$$

$$h_U = \mathrm{Hom}(U, -) \qquad \mathrm{Hom}_{\mathrm{ord}}(h_U, F) = F(U)$$
$$\shortparallel$$
$$\mathrm{Hom}_{\mathrm{Pro}}(F, h_U)$$

$$F = \underline{\mathrm{Hom}}_{\mathrm{ProVect}}(V, -) \qquad F(U) = \mathrm{colim}\, \underline{\mathrm{Hom}}(V_i, U)$$
$$G = \underline{\mathrm{Hom}}_{\mathrm{ProVect}}(W, -) \qquad G(U) = \mathrm{colim}\, \underline{\mathrm{Hom}}(W_j, \;)$$

$$\rightsquigarrow (F \tilde{\otimes} G)(U) = (\mathrm{colim}\, \underline{\mathrm{Hom}}(V_i, U)) \otimes (\mathrm{colim}\, \underline{\mathrm{Hom}}(W_j, U))$$
$$= \mathrm{colim}_{i,j}\, \underline{\mathrm{Hom}}(V_i, U) \otimes \underline{\mathrm{Hom}}(W_j, U)$$

$$(\mathrm{End}(\mathrm{Vect}), \circ) \otimes (\mathrm{End}(\mathrm{Vect}), \circ) \xrightarrow{\overset{!}{\otimes}} (\mathrm{End}(\mathrm{Vect}), \circ)$$

$$\mathrm{Hom}_{\mathrm{Alg}^{\mathrm{lax}}}(\mathcal{E}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{lax}}}(\mathcal{E} \bar{\otimes} \mathcal{C}, \mathcal{D})$$

$$(\mathrm{ProVect}, \vec{\otimes}) \otimes (\mathrm{ProVect}, \vec{\otimes}) \otimes \mathrm{Vect} \longrightarrow \mathrm{Vect}$$

$$A \otimes B \otimes C \longmapsto \mathrm{Hom}^{\mathrm{cont}}(A \overset{!}{\otimes} B, C)$$

$$(A_1 \vec{\otimes} A_2) \otimes (B_1 \vec{\otimes} B_2) \otimes (C_1 \otimes C_2)$$

Cor • $\overset{!}{\otimes}$ algebras are naturally $\vec{\otimes}$ algebras

• If $A, B \in Alg^{\vec{\otimes}}$ $A \overset{!}{\otimes} B \in Alg^{\vec{\otimes}}$

On modules

We have $A\text{-mod}_{top} \times B\text{-mod}_{top} \xrightarrow{\overset{!}{\otimes}} (A \overset{!}{\otimes} B)\text{-mod}_{top}$

by general nonsense which induces

$$A\text{-mod}_{naive} \times B\text{-mod}_{naive} \longrightarrow (A \overset{!}{\otimes} B)\text{-mod}_{naive}$$

! Achtung ! When looking at usual algebras (connective)

$$A\text{-mod} \otimes B\text{-mod} \simeq (A \otimes B)\text{-mod}.$$

Rmks $\overset{!}{\otimes}$ product of $\vec{\otimes}$-modules is an $\vec{\otimes}$ module

$$\begin{matrix} A \vec{\otimes} M \to M \\ B \vec{\otimes} N \to N \end{matrix} \quad \rightsquigarrow \quad (A \overset{!}{\otimes} B) \vec{\otimes} (M \overset{!}{\otimes} N) \longrightarrow$$

Renormalization → **Idea**: Bootstrap from finite dimensional situations

① On ∞-Categories it is the choice of a subcat of compacts $C^\circ \subseteq C^+$, $\text{Ind}(C^\circ)^+ \xrightarrow{\sim} C^+$ } Roughly is the work of IndCoh

② Renormalization of H-mod

$$H\text{-mod}_{weak} = \text{Rep}(H)\text{-mod} \qquad \text{Rep}(H) = \text{Ind}(\text{Rep}(H)^c)$$

$$H\text{-mod}_{weak,naive} = \text{QCoh}(H)\text{-mod}$$

$$\text{Vect} \in (\text{Rep}(H)_{naive}, \text{QCoh}(H))\text{-mod}$$

$$\uparrow$$

$$(\text{Rep}(H), \text{QCoh}(H))$$

$$\rightsquigarrow \quad H\text{-mod}_{weak} \longrightarrow H\text{-mod}_{weak,naive}$$

$$\mathcal{C} \longmapsto \text{Vect} \otimes_{\text{Rep}(H)} \mathcal{C} \in \text{QCoh}(H)\text{-mod}$$

Ntz. For $\mathcal{D} \in H\text{-mod}_{weak}$ we think $\mathcal{D} = \mathcal{C}^{H,w}$ for $\mathcal{C} \in H\text{-mod}_{weak}$

$$\begin{array}{ccccc} \text{Rep}(H)\text{-mod} = & H\text{-mod}_{weak} & \xrightarrow{\text{Vect} \otimes_{\text{Rep}(H)} -} & H\text{-mod}_{weak,naive} & \\ \text{Oblv} \searrow & \downarrow & \nearrow & \downarrow & (-)^{H,w,naive} \\ & \text{DGCat}_{cont} & \longrightarrow & \text{DGCat}_{cont} & \end{array}$$

- Vect dualizable over $\text{Rep}(H)$ $\Rightarrow$ $\exists$ L, R adjoints of

$$H\text{-mod}_{weak} \rightleftarrows H\text{-mod}_{weak,naive}$$

Like we usually look at connective algebras

$\in \mathrm{ProVect}^{\leq 0}$

We look at connective $\vec{\otimes}$ algebras (left t-exact)

Remind $A\text{-mod}_{naive}$ has a canonical t-structure for which

Fact $\mathrm{Oblv} : A\text{-mod}_{naive} \longrightarrow \mathrm{Vect}$ is t-exact

This structure is right complete since Oblv is continuous

right complete = $\mathrm{colim}\, \tau^{\leq n} X \xrightarrow{\sim} X$

Oblv conservative + Vect right complete

Convergence $V \in \mathrm{ProVect}$ $\quad \hat{V} = \lim_n \tau^{\geq -n} V \in \mathrm{ProVect}$

(Convergent completion)

def We say V is convergent if $V \to \hat{V}$ is iso

rmk V is convergent if and only if it lies in $\mathrm{Pro}(\mathrm{Vect}^+) \subseteq \mathrm{Pro}(\mathrm{Vect})$ (Not complete)

equivalently, $F_V : \mathrm{Vect} \longrightarrow \mathrm{Vect}$ is left Kan extended from

$$F_V : \mathrm{Vect}^+ \longrightarrow \mathrm{Vect}$$

Why?

$\mathcal{C} \subseteq \mathcal{D}$ $\qquad \mathrm{Pro}(\mathcal{C}) \xrightarrow{\varphi} \mathrm{Pro}(\mathcal{D})$

$\qquad \cap \qquad\qquad \cap$

$\mathrm{Hom}(\mathcal{C}, \mathcal{S})^{op} \xrightarrow{?} \mathrm{Hom}(\mathcal{D}, \mathcal{S})^{op}$

$\varphi(X)$ is the left Kan extension $\mathcal{C} \xrightarrow{X} \mathcal{S}$, $\mathcal{C} \subseteq \mathcal{D}$, $\mathrm{Lan}\, X = \varphi(X) : \mathcal{D} \to \mathcal{S}$

$X = \lim X_i$ ($X_i \in \mathcal{C}$) $\longmapsto \lim h_{X_i} = \mathrm{Map}(X, -)$, $\mathcal{D} \to \mathcal{S}$

$\varphi(X)(d) = \mathrm{Map}(\lim h_{X_i}, d)$

$= \mathrm{colim}\, \mathrm{Map}(X_i, d)$

prop The category of connective, convergent $\vec{\otimes}$ algebras is op-equivalent to the category of left t-exact, (accessible) comonads $\mathrm{Vect}^+ \to \mathrm{Vect}^+$

rmk If $A \in \mathrm{Alg}^{\vec{\otimes}}$ is connective then $\hat{A} \in \mathrm{Alg}^{\vec{\otimes}}$ is connective
and $\hat{A}\text{-mod}^{+}_{\text{naive}} \xrightarrow{\sim} A\text{-mod}^{+}_{\text{naive}}$

pf Connectivity easy since $\tau^{\leq 0}(\tau^{\geq n} A) = \tau^{\geq n}(\tau^{\leq 0} A)$ + $\tau^{\leq 0}$ right adjoint

1) $\vec{\otimes}$ is continuous in the first variable
there is a natural transformation $A \vec{\otimes} \lim B_j \longrightarrow \lim A \vec{\otimes} B_j$
so a limit of $\vec{\otimes}$ algebras is naturally a $\vec{\otimes}$ algebra

2) $\tau^{\geq n}$ is ~~limit preserving~~ and for objects in Vect there is
a natural transformation

$$\tau^{\geq n} X \otimes \tau^{\geq n} Y \leftarrow \tau^{\geq 2n}(X \otimes Y)$$

(n)

$\tau^{\geq n} X = \dots \to 0 \to \text{coker} \to X^{n+1} \to X^{n+2} \to \dots$

$0 \geq k > 2n$

$$(\tau^{\geq n} X \otimes \tau^{\geq n} Y)^k = \bigoplus_{i > n,\, j > n,\, i+j=k} X^i \otimes Y^j \oplus \text{coker} \otimes Y^{k-n} \oplus X^{k-n} \otimes \text{coker}$$

$\oplus\, \text{coker} \otimes \text{coker}$ $\quad k = 2n$

$$\tau^{\geq 2n}(X \otimes Y)^k = \bigoplus_{i+j=k} X^i \otimes Y^j \quad k > 2n$$

$\text{coker}\left(\bigoplus_{n-1} X^i \otimes Y^j \to \bigoplus_{n} X^i \otimes Y^j\right)$

$\mathrm{Hom}[X[n], Y]^k = \prod \mathrm{Hom}(X^{i+1}, Y^{i+k})$

$$\begin{array}{ccc} \tau^{\geq n} A \,\vec{\otimes}\, \tau^{\geq n} A & \longrightarrow & \tau^{\geq 2n} A \\ \uparrow & & \uparrow \\ A \,\vec{\otimes}\, A & \longrightarrow & A \end{array}$$

There is a natural

Remind $[1]$ on $\mathrm{End}(\mathrm{Vect})^{op}$ acts
$\mathrm{Hom}_{\mathrm{ProVect}}(X[1], U)$
$\|$
$\mathrm{Hom}_{\mathrm{ProVect}}(X, U[-1])$ as $\_ \circ [-1]$ on End

3) $A : \mathrm{Vect} \to \mathrm{Vect}$ $\quad \tau^{\geq n} A = \tau^{\geq n} \circ A$

$$\tau^{\geq n} = [-n] \circ \tau^{\geq 0} \circ [n]$$

restrict to $\mathrm{Vect}^{\geq 0}$ + Lan $\quad$ $\_ \circ [-n]$

(A connective)

- $\tau^{\geq 0}$ fa $|_{Vect^{\geq 0}}$ e Lan

- $\tau^{\geq n}$ fa $\tau^{\geq n} : \mathrm{Pr}\,Vect \to \mathrm{Pr}(Vect^{\geq n})$

$(\tau^{\geq n} T)(V) = ((\tau^{\geq n} T)(V[-n]))$

A (connective) $A^{\geq n} : Vect^{\geq n} \to Vect$

claim $A \to S^{\in \mathrm{Pr}\,Vect^{\geq n}}$ = $\mathrm{Hom}_{(Vect^{\geq n} \to Vect)^{\geq n}}(A|_{Vect^{\geq n}}, S)$

$\quad\;\; \mathrm{Lan}(S|_{Vect^{\geq n}})$

$$\mathrm{Map}(\mathrm{Lan}\, S, A) = \mathrm{Map}(S|_{\mathcal{C}}, A|_{\mathcal{C}})$$

- Lan of rest of comonads is a comonad

$\mathrm{Lan}\, T \to \mathrm{Lan}\, T \circ \mathrm{Lan}\, T$

$T|_{-} \longrightarrow \mathrm{Lan}\, T \circ \mathrm{Lan}\, T|_{-}$

$\quad T^2$ ← Se con pure da la sotto cat è preservata

$DGAlg^{\leq 0} \overset{?}{\Longrightarrow} \tau^{\geq -n} A \in DGAlg^{\leq 0}$ $\quad A \to \tau^{\geq -n} A$

$\tau^{\geq -n} A \vec{\otimes} \tau^{\geq -n} A \overset{?}{\to} \tau^{\geq -n} A \leftarrow \tau^{\geq -n} m$

$\searrow \tau^{\geq -n}(\tau^{\geq -n} A \otimes \tau^{\geq -n} A) \overset{!}{\simeq} \tau^{\geq -n}(A \vec{\otimes} A)$

Rmk A connective $\vec{\otimes}$ Alg. Then

$$\hat{A}\text{-mod}^{+}_{naive} \xrightarrow{\sim} A\text{-mod}^{+}_{naive}$$

$\tau^{\geq -n}\hat{A} \overset{?}{=} \tau^{\geq -n} A$ ← is a left adjoint

as comonads it is clear though

Better $\forall n$

Almost definition

1) $\hat{A}\text{-mod}^{\geq n}_{naive} = \tau^{\geq n}\hat{A}\text{-mod}^{\geq n}_{naive} = A\text{-mod}^{\geq c}_{naive}$

clear.

Rmk colim of comonads

$$(\operatorname{colim} \tau^{\geq -n} A)(V) = \operatorname*{colim}_n \tau^{\geq -n} A(V)$$

$$= \operatorname*{colim}_n \operatorname*{colim}_{U \in \mathrm{Vect}^{\geq -n}/V} A(U)$$

⇝ If $V \in \mathrm{Vect}^{\geq -k}$ $\quad \mathrm{Vect}^{\geq -n}/V \xleftarrow{\sim} \mathrm{Vect}^{\geq -k}/V$

⇝ $(\operatorname{colim} \tau^{\geq -n} A)|_{\mathrm{Vect}^{\geq -k}} = A|_{\mathrm{Vect}^{\geq -k}}$

def A renormalization datum for a connective $\otimes^?$ algebra A is a DG category $A\text{-mod}_{ren}$ equipped with a t-structure and an equivalence

$\rho : A\text{-mod}_{ren}^{+} \longrightarrow A\text{-mod}_{naive}$ such that

1) ρ is t-exact
2) $A\text{-mod}_{ren}$ is compactly generated by generators lying in $A\text{-mod}_{ren}^{+}$

3) The t-structure on $A\text{-mod}_{ren}$ is cpt gen

i.e. $G \in A\text{-mod}_{ren}^{\geq 0}$ iff

$$\mathrm{Hom}_{A\text{-mod}_{ren}}(F, G) = 0 \quad \forall\ F \in A\text{-mod}_{ren}^{\leq 0} \text{ compact.}$$

ex

1) IndCoh. Assume A is classical of finite type

Defn $A\text{-mod}_{ren} = \mathrm{Ind}(\mathrm{Coh}(\mathrm{Spec}\, A))$

$A\text{-mod} \simeq$ (?) ; A smooth $\Rightarrow A\text{-mod}_{ren}$

2) Assume A is left coherent (finitely gen ideals are also finitely presented)

(this includes poly algs in an infinite number of variables) and their localizations

$A\text{-mod}_{coh} = \{$ finitely presented cohomologies, bounded $\}$

$\mathrm{Ind}(A\text{-mod}_{coh}) = A\text{-mod}_{ren}$

$A = \mathcal{O}_G$ affine (pro) group scheme

$\Longrightarrow A\text{-mod}_{ren} = A\text{-mod}.$

3) $A_i \in Alg(\mathrm{Vect}^{c\sigma})$ proj system $\lim A_i \in Alg^{\overset{!}{\otimes}} \to Alg^{\overrightarrow{\otimes}}$

↖ left coherent

Assume $A_j \twoheadrightarrow A_i$ is surjective with finitely gen (and so presented) kernel

Then $A_j\text{-mod}_{coh} \to A_i\text{-mod}_{coh}$

$\rightsquigarrow$ $A_j\text{-mod}_{ren} \to A_i\text{-mod}_{ren}$

$A\text{-mod}_{ren} = \operatorname{colim} A_i\text{-mod}_{ren}$ is a renormalization datum for A.

Fact In the classical case $A\text{-mod}_{ren}$ has a nice t-structure whose heart is equivalent to the category of discrete A-modules

Nt: Let $A\text{-mod}^c_{ren} \subseteq A\text{-mod}^+_{naive}$ the category of cpt objects in $A\text{-mod}_{ren}$.

Consider $A\text{-mod}^+_{naive} \otimes B\text{-mod}^+_{naive} \longrightarrow (A \overset{!}{\otimes} B)\text{-mod}^+_{naive}$

idempotent completion ↙

And let $(A \overset{!}{\otimes} B)\text{-mod}^c_{ren}$ the Karoubi envelope of the image

$$A\text{-mod}^c_{ren} \times B\text{-mod}^c_{ren} \longrightarrow (A \overset{!}{\otimes} B)\text{-mod}^+_{naive}$$

Thm $(A \overset{!}{\otimes} B)\text{-mod}^c_{ren}$ defines a renormalization datum for $A \overset{!}{\otimes} B$ for which the ind-extension of

$$A\text{-mod}^c_{ren} \times B\text{-mod}^c_{ren} \longrightarrow (A \overset{!}{\otimes} B)\text{-mod}^c_{ren}$$

Induces an isomorphism

$$A\text{-mod}_{ren} \otimes B\text{-mod}_{ren} \longrightarrow (A \overset{!}{\otimes} B)\text{-mod}_{ren}$$