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Description of the center of the enveloping algebra of an affine Kac-Moody algebra

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CANDIDATE
Luca Casarin

ADVISOR
Andrea Maffei
Università di Pisa

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Chapter 1

Introduction

In this first chapter of the thesis we are going to define the main objects of interest and our main goal which is the description of the center of an enveloping algebra associated to an affine Kac-Moody algebra.

A very important class of groups is the one of loop groups. Given G a connected algebraic reductive group over the complex numbers, we consider its loop group LG. It may be viewed as the group formed by the maps from the pointed formal disc $D^* \to G$ and it is defined as the group functor

$$LG : \mathbb{C}Alg \to Grp \qquad LG(R) = G(R((t)))$$

Where R stands for an arbitrary $\mathbb{C}$ algebra and $R((t))$ is the algebra of Laurent series with coefficients in R. If the group is affine then it can be proved that LG is an ind-scheme.

In order to study the representation theory for loop groups it is natural to investigate the representation theory for their Lie algebras.

Let $\mathfrak{g} = Lie(G)$, since the group we are considering is affine we have

$$\mathfrak{g}(R) = \mathfrak{g} \otimes R = ker(G(R[\epsilon] \to G(R))$$

We easily deduce that that $Lie(LG) = \mathfrak{g}((t)) =: L\mathfrak{g}$ or more precisely

$$Lie(LG)(R) =: L\mathfrak{g}(R) = \mathfrak{g}(R((t))) = \mathfrak{g} \otimes R((t))$$

We are interested in the $\mathbb{C}$ points of this Lie algebra which we will continue denotig by $L\mathfrak{g}$ or $\mathfrak{g}((t))$. Note that the bracket here is simply determined by the bracket on $\mathfrak{g}$, extended by $\mathbb{C}((t))$-linearity. In the study of the latter algebra it is particularly interesting to study the representation theory of its central one dimensional extension $\hat{\mathfrak{g}}_k$. The following proposition, that we will not prove, holds.

Proposition 1.0.1. *Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers. Then the second cohomology group $H^2(L\mathfrak{g}, \mathbb{C})$ is one dimensional and generated by the cocycle*

$$c_{\mathfrak{g}}(X \otimes f, Y \otimes g) = \kappa_{\mathfrak{g}}(X, Y) Res_{t=0}(fdg)$$

where $\kappa_{\mathfrak{g}}$ is the killing form on $\mathfrak{g}$ while f and g are two Laurent series with coefficients in $\mathbb{C}$. Since this group classifies isomorphism classes of central one dimensional extensions of $L\mathfrak{g}$ we obtain that they are in correspondence with $\mathbb{C}$.

Definition 1.0.1. Let $\mathfrak{g}$ be a simple complex Lie algebra and $k \in \mathbb{C}$ a complex number. We define the affine Kac-Moody algebra (or simply the affine algebra) $\hat{\mathfrak{g}}_k$ to be the central extension of $L\mathfrak{g}$ determined by the cocycle $kc_{\mathfrak{g}}$ where $c_{\mathfrak{g}}$ is the cocycle of the preceding proposition. It satisfies the natural exact sequence

$$0 \to \mathbf{1}\mathbb{C} \to \hat{\mathfrak{g}}_k \to L\mathfrak{g} \to 0$$

As a vector space we have

$$\hat{\mathfrak{g}}_k = L\mathfrak{g} \oplus \mathbf{1}\mathbb{C}$$

while the bracket is defined through the formula

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg - k\kappa_{\mathfrak{g}}(X, Y)\mathrm{Res}_{t=0}(fdg)\mathbf{1} \tag{1.1}$$

We will sometimes write for convenience $k\kappa_{\mathfrak{g}} = \kappa$, this is of course an associative form on $\mathfrak{g}$ since it is a scalar multiple of the Killing form. This construction is available also for a general Lie algebra $\mathfrak{g}$ with an associative symmetric form κ, the hypothesis of $\mathfrak{g}$ being simple bounds us to consider only multiples of the Killing form.

Remark 1.0.1. The affine algebra $\hat{\mathfrak{g}}_k$ posses two additional natural structures.

- It is in a natural way a topological vector space, with the topology induced by the vector subspaces $\mathfrak{g} \otimes t^N\mathfrak{g}[[t]]$ with $N \in \mathbb{Z}$;
- It carries a natural $\mathrm{Aut}\,\mathcal{O}$ module structure. Indeed since the bracket on $L\mathfrak{g}$ is $\mathbb{C}((t))$ linear $\mathrm{Aut}\,\mathcal{O}$ naturally acts on it along the factor $\mathbb{C}((t))$. On the other hand the cocycle defining the affine algebra κ is invariant under such an action since the residue $\mathrm{Res}_{t=0}fdg$ is invariant under automorphisms. The action of $\mathrm{Aut}\,\mathcal{O}$ extends therefore to the affine algebra $\hat{\mathfrak{g}}_k$, it naturally induces an action of its Lie algebra $\mathrm{Der}\,\mathcal{O}$.

We are interested in the the category of $\hat{\mathfrak{g}}_k$ representations which are smooth (i.e. $t^N\mathfrak{g}[[t]]$ acts locally trivially) and in which $\mathbf{1}$ acts as the identity.

Definition 1.0.2. A module V over $\hat{\mathfrak{g}}_k$ is said to be *smooth* if $\mathbf{1}$ acts as the identity and if, for every $v \in V$, there exists a sufficiently large natural number N such that the subalgebra $t^N\mathfrak{g}[[t]]$ annihilates v:

$$t^N\mathfrak{g}[[t]] \cdot v = 0$$

Our goal is to describe the center of the category of smooth $\hat{\mathfrak{g}}_k$ modules, we will see that this is the same as computing the center of a certain associative algebra. This is of crucial importance in studying the representation theory of $\hat{\mathfrak{g}}_k$. Every central element acts intertwining with the action of $\hat{\mathfrak{g}}_k$ on any given module. In particular on irreducible finite dimensional representations the center acts like the multiplication of a character. The study of the center then permits for instance to distinguish finite dimensional representation through the associated character.

In the following paragraph we briefly recall what the center of an abelian category is and explain that it essentially coincides with the notion of the center of an associative algebra when our category is the abelian category of modules over an algebra.

1.1 The center of an abelian category

Recall the definition of the center of an abelian category, which is by definition the set of endomorphisms of the identity functor.

Definition 1.1.1. Let C be an abelian category. If the natural transformations from the identity functor to itself form a set we call it $Z(C)$ the center of the category. An element of the center of C is therefore a collection of endomorphisms e_M for each object $M \in C$ such that for any morphism $\varphi : M \to N$ we have $e_N \circ \varphi = \varphi \circ e_M$.

It is easy to see that, when it is a set, the center of an abelian category is an abelian ring, if moreover the category is k-linear for some field F the center is easily seen to be a F commutative algebra. Indeed, given two elements $e, f \in Z(C)$, since C is abelian, it is easy to see that $(e+f)_M := e_M + f_M$ is still an endomorphism of the identity functor and the same is true for the composition $(e \cdot f)_M := e_M \circ f_M$. Using the fact that e, f are central and that C is abelian one can easily prove that these operations define the structure of a commutative ring. The definition of scalar multiplication is done similarly.

Moreover, by definition, we have a natural action of $Z(C)$ on any object $M \in C$, meaning a morphism of algebras $Z(C) \to \mathrm{End}_C(M, M)$. We consider now $\mathbb{C}$ linear categories, so that $Z(C)$ is a $\mathbb{C}$ algebra.

Consider the scheme $S = \mathrm{Spec}\,(Z(C))$ and a closed point $x \in S$ which gives a homomorphism $\rho_x : Z(C) \to \mathbb{C}$. We may consider the fullsubcategory C_x of C whose objects are the objects in C for which $Z(C)$ acts accordingly to the character ρ_x.

As an example consider $\mathfrak{g}$ a simple finite dimensional Lie algebra over $\mathbb{C}$. The abelian category of $\mathfrak{g}$-modules is equivalent to the category of left $U(\mathfrak{g})$ modules.

We now describe the center of this category: we claim that it equals the center $Z(\mathfrak{g})$ of the associative algebra $U(\mathfrak{g})$.

Indeed any central element defines, through its action on modules, a central element of the category and we obtain a morphism $Z(\mathfrak{g}) \to Z(C)$. This is easily seen to be injective since for any element $z \in Z(\mathfrak{g})$ its action on the module $U(\mathfrak{g})$ is not trivial. On the other hand consider an element $e \in Z(C)$, consider its action on the module $U(\mathfrak{g})$ and in particular the element $e(1)$. Since e is central and right multiplication on $U(\mathfrak{g})$ is an endomorphism of $\mathfrak{g}$-modules $e(1)$ satisfies $e(x) = e(1 \cdot x) = e(1) \cdot x$ but by definition e must commute with the action of $U(\mathfrak{g})$ so we also have that $e(f) = e(f \cdot 1) = f \cdot e(1)$ we deduce from this that $e(1) \in Z(\mathfrak{g})$ and it is quite easy to see that $e(1)$ determines e on all modules.

In this particular case if we focus on finite dimensional representations we have that the category is 'generated' by irreducible representations, on which the center acts by mulitplication by scalars. The category of $\mathfrak{g}$ modules of finite dimension is generated by the full subcategories $(\mathfrak{g}\text{-mod})_x$ defined above, for x a closed point of $\mathrm{Spec}\, Z(\mathfrak{g})$. In this sense $\mathfrak{g}$-mod may be thought of as 'fibering' over $\mathrm{Spec}\, Z(\mathfrak{g})$.

1.2 The completed enveloping algebra

Analogously to the finite dimensional case, in which given a Lie algebra $\mathfrak{g}$ we consider its enveloping algebra $U(\mathfrak{g})$ we try to define an associative algebra whose left modules correspond to smooth representations of $\hat{\mathfrak{g}}_k$. It turns out that actually we must consider a topological associative algebra and its continuous modules. It is clear that the classical enveloping algebra $U(\hat{\mathfrak{g}}_k)$ does not suffice for this purpose, but a slight modification of it will do the trick.

The condition of $\mathbf{1}$ acting as the identity may be imposed by considering first the algebra $U_k(\hat{\mathfrak{g}}) := U(\hat{\mathfrak{g}}_k)/(1-\mathbf{1})$. Consider next the topology on $U_k(\hat{\mathfrak{g}})$ induced by the left ideals $I_n := U(\hat{\mathfrak{g}}_k)(t^N\mathfrak{g}[[t]])$. It may be checked that the product on the enveloping algebra is continuous for this topology and therefore may be extended to a product on the completed enveloping algebra

$$\widetilde{U}_\kappa(\hat{\mathfrak{g}}) = \varprojlim_n U_k(\hat{\mathfrak{g}})/I_n$$

Note that this is a limit of $U_k(\hat{\mathfrak{g}})$ modules and not a limit of algebras. Anyway, by the continuity of the bracket, one may extend it to $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. This is a complete topological algebra.

Proposition 1.2.1. *The category of smooth $\hat{\mathfrak{g}}_k$ modules is equivalent to the category of smooth left $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ modules:*

$$\hat{\mathfrak{g}}_k\text{- } mod_{\mathrm{smooth}} = \widetilde{U}_\kappa(\hat{\mathfrak{g}})\text{- } mod_{\mathrm{cont}}$$

Proof. We prove that to give the structure of a smooth $\hat{\mathfrak{g}}_k$ module on a $\mathbb{C}$ vector space V is equivalent to give a structure of left $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ smooth module. Given a smooth $\hat{\mathfrak{g}}_k$ module M it gains naturally a structure $U(\hat{\mathfrak{g}}_k)/(1-\mathbf{1})$ module (since $\mathbf{1}$ acts as the identity on M). In addition the smoothness condition on $\hat{\mathfrak{g}}_k$ implies that M with the discrete topology is a continuous $U(\hat{\mathfrak{g}}_k)/(1-\mathbf{1})$ module for the topology induced by the ideals I_n and hence a $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ continuous module.

On the other hand consider a continuous $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ module M. The continuous homomorphism of Lie algebras

$$\hat{\mathfrak{g}}_k \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

induces a structure of $\hat{\mathfrak{g}}_k$ module on M. The verification that the above constructions are one the inverse of the other is trivial. ☐

Notice that while $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is a left module over itself, it is not a smooth module. Indeed a smooth module over a topological algebra is to be intended as a vector space equipped with the discrete topology and with a continuous action for this topology. $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ does not satisfy this continuity condition.

Fortunately for us, the quotients by the left ideals I_n, which are always defined to be as the left ideals generated by $t^N\mathfrak{g}[[t]]$, really are continuous modules.

Proposition 1.2.2. *The center $Z_\kappa(\hat{\mathfrak{g}})$ of the category of $\hat{\mathfrak{g}}_k$ smooth modules equals the center of the completed enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.*

Proof. It is clear that every element of $Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}}))$ defines an element of $Z_\kappa(\hat{\mathfrak{g}})$ and that we obtain a morphism

$$Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}})) \to Z_\kappa(\hat{\mathfrak{g}})$$

We now prove injectivity and surjectivity.

As injectivity is concerned let $x \in Z(\widetilde{U}_\kappa(\hat{\mathfrak{g}}))$ be a central element which is sent to 0 through the above morphism. Consider its action on the smooth modules $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ which are all trivial by hypothesis. Then we have that $x = x \cdot 1 \in I_n$ for all $n \in \mathbb{Z}_{\geq 0}$, it is easy to check though that $\cap_n I_n = 0$ and therefore $x = 0$.

On the other hand consider an element $e \in Z_\kappa(\hat{\mathfrak{g}})$ in the center of the category. Consider also $e_n := e(1) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$: this collection provides us with a unique element $e(1) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ since by hypothesis e is central and the projections $\pi : \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_m$ are morphism of representations. As in the case of the example in the previous paragraph it is not difficult to prove that $e(1)$ is central and that determines completely e.

□

Thanks to the above proposition we will call the center of the enveloping algebra $Z_\kappa(\hat{\mathfrak{g}})$. The goal of the thesis is the description of this center as $k \in \mathbb{C}$ varies.

It turns out that $Z_\kappa(\hat{\mathfrak{g}})$ is trivial almost for all complex numbers except for a specific value of k we will focus on the description of the center in this non-trivial case.

We are interested in a geometric description: as we noticed before the affine algebra $\hat{\mathfrak{g}}_k$ carries a natural $\operatorname{Aut}\mathcal{O}$ action. This action extends to $U(\hat{\mathfrak{g}}_k)$ and ultimately to $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ since the subalgebras $t^N\mathfrak{g}[[t]]$ are actually invariant under the $\operatorname{Aut}\mathcal{O}$ action. By geometric we mean a description which takes into account this action and states an isomorphism between $Z_\kappa(\hat{\mathfrak{g}})$ and some algebra of functions over a space related to the disc.

We will focus on proving the second part of the following theorem due to Feigin and Frenkel, following [Fre07].

Theorem 1.2.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The following description for the center of the completed enveloping algebra of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ holds:*

- *If $\kappa \neq -\frac{1}{2}\kappa_{\mathfrak{g}}$ the center $Z_\kappa(\hat{\mathfrak{g}})$ is trivial (i.e. isomorphic to $\mathbb{C}$);*
- *If $\kappa = -\frac{1}{2}\kappa_{\mathfrak{g}}$ (we denote $\kappa_c := -\frac{1}{2}\kappa_{\mathfrak{g}}$ the critical value) then*
    1. *The center $Z_{\kappa_c}(\hat{\mathfrak{g}})$ contains a free polynomial algebra in an infinite number of variables which topologically generates it. In particular it is isomorphic to the completion of a polynomial algebra in an infinite number of varibles;*
    2. *There exists a natural $(\operatorname{Aut}\mathcal{O}, \operatorname{Der}\mathcal{O})$-equivariant isomorphism*

$$Z_{\kappa_c}(\hat{\mathfrak{g}}) = \mathbb{C}[\mathrm{Op}_{^L G}(D^*)]$$

with the space of functions on the space of $^L G$-Opers on the punctured disc D^*.

Here $^L G$ stands for the Langlands dual group of G while we postpone the definition of the space of Opers to the last section. We remark that actually part 2 of the second statement implies part 1: it is not difficult to algebraically describe $\mathbb{C}[\mathrm{Op}_{^L G}(D^*)]$ as the completion of a polynomial algebra. We anyway state these results separately to remark the algebraic nature of the first statement and the geometric nature of the second one.

1.3 Strategy of the proof and organization of this work

In this section we give a brief outlook of how the proof of Theorem 1.2.1 will be carried out throughout the thesis. We will make references to various constructions, the reader may jump through the chapters in order to take a look at the various definitions.

The entire thesis revolves around chapters 2-8 of Edward Frenkel's book 'Langlands Correspondence for Loop Groups' [Fre07]. Our goal is mainly to follow Frenkel's proof of Theorem 1.2.1, paying specific attention and writing down all the details and the computations that do not appear in the book. We will, from time to time, omit some proofs and refer to the book, especially for the ones that are already written in detail.

Throughout this discussion we assume that any simple Lie algebra $\mathfrak{g}$ considered is equipped with a chosen maximal toral subalgebra $\mathfrak{h}$ and a chosen basis for the root system Φ relative to $\mathfrak{h}$ which induces a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. We call $\mathfrak{b}_+ := \mathfrak{n}_+ \oplus \mathfrak{h}$ the upper Borel subalgebra and $\mathfrak{b}_- := \mathfrak{n}_- \oplus \mathfrak{h}$ the lower Borel subalgebra.

The vertex algebra $V_\kappa(\mathfrak{g})$

We start by considering a different algebraic object from the completed enveloping algebra: the vertex algebra $V_\kappa(\mathfrak{g})$. Vertex algebras are algebraic structures which arise quite naturally in this context. A vertex algebra V is essentially a vector space equipped with infinitely many products, indexed by the natural numbers $\mathbb{Z}$ and denoted by $A_n B$ for $A, B \in V$, which satisfy certain axioms. We are able to attach to any affine algebra $\hat{\mathfrak{g}}_\kappa$ a vertex algebra closely related to it, we call it $V_\kappa(\mathfrak{g})$: the vacuum Verma module. It is a $\hat{\mathfrak{g}}_\kappa$ module constructed as follows

$$V_\kappa(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_\kappa} \mathbb{C}|0\rangle$$

where $\mathbb{C}|0\rangle$ is the trivial $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ module where $\mathfrak{g}[[t]]$ acts like 0 and $\mathbf{1}$ acts like the identity.

The structure of vertex algebra is induced from the structure of $\hat{\mathfrak{g}}_\kappa$ module. This object is way easier to study than the complete enveloping algebra: first it has a simpler description as a $\hat{\mathfrak{g}}_\kappa$ module, and second the structure of Vertex algebra allows to simplify a lot of calculations.

This is all done in chapter 3: we introduce the basic theory of vertex algebra and define both $V_\kappa(\mathfrak{g})$ and the Virasoro vertex algebra, the latter is a very important example of vertex algebra as well as it is crucial for our goals.

There is a natural notion of **center** of a vertex algebra and it turns out that the center of $V_\kappa(\mathfrak{g})$, which we call $\zeta_\kappa(\mathfrak{g})$, coincides with the space of $\mathfrak{g}[[t]]$ invariants

$$\zeta_\kappa(\mathfrak{g}) = V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$$

it has a natural structure of commutative $\mathbb{C}$ algebra induced by the product $A \cdot B = A_{-1} B$.

It is quite natural to start studying the center of the vertex algebra in order to obtain some information about the center of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. In order to describe it we consider the natural filtration on $V_\kappa(\mathfrak{g})$ induced by the classical PBW filtration. The associated graded space has a natural structure of commutative algebra. We are able to prove an isomorphism

$$\mathrm{gr}\, V_\kappa(\mathfrak{g}) \simeq \mathbb{C}\left[\mathfrak{g}^*[[t]]\right]$$

with the ring of regular functions on $\mathfrak{g}^*[[t]]$. This an algebra isomorphism which intertwines with the action of $\mathrm{Aut}\,\mathcal{O}$ on both spaces. From this isomorphism we obtain a natural embedding

$$\mathrm{gr}\left(V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}\right) = \mathrm{gr}\,\zeta_\kappa(\mathfrak{g}) \hookrightarrow \mathbb{C}\left[\mathfrak{g}^*[[t]]\right]^{\mathfrak{g}[[t]]} =: \mathrm{Inv}\ \mathfrak{g}^*[[t]]$$

Note that in contrast with the finite dimensional case it is far from being obvious and it is actually false in general that this embedding is an isomorphism.

We focus on the proof of the following theorem which is the vertex algebra analogue of theorem 1.2.1:

Theorem 1.3.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $V_\kappa(\mathfrak{g})$ the vertex algebra associated to the affine Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ of level κ. Then the following hold:*

- *If $\kappa \neq \kappa_c$ the center $\zeta_\kappa(\mathfrak{g})$ is trivial (i.e. spanned by $|0\rangle$);*
- *If $\kappa = \kappa_c$ we have*
  1. *The immersion $\operatorname{gr} \zeta_{\kappa_c}(\mathfrak{g}) \to \operatorname{Inv} \mathfrak{g}^*[[t]]$ is an isomorphism and therefore $\zeta(\mathfrak{g})$ is a free polynomial algebra in an infinite number of variables;*
  2. *The center $\zeta(\mathfrak{g})$ of $V_{\kappa_c}(\mathfrak{g})$ is isomorphic in an $(\operatorname{Aut} \mathcal{O}, \operatorname{Der} \mathcal{O})$ equivariant way the the algebra of functions on the space of ${}^L G$ Opers on the formal disc D.*

$$\zeta(\mathfrak{g}) = \mathbb{C}[\mathrm{Op}_{{}^L G}(D)]$$

From $V_\kappa(\mathfrak{g})$ to $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$

Before diving into the proof of theorem 1.3.1 we explore in more detail the connection between $V_\kappa(\mathfrak{g})$ and $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. We define a functor which associates to any vertex algebra V a Lie algebra $U(V)$. The latter is spanned by elements of the form $A_{[n]}$ wit $A \in V$ and should be understood as a formal analogue of the Lie subalgebra of $\operatorname{End} V$ spanned by the endomorphisms induced by the various n-products $A_n(\cdot) \in \operatorname{End} V$. The bracket on $U(V)$ is defined following the vertex algebra axioms. This construction is the key ingredient to pass from the vertex algebra $V_\kappa(\mathfrak{g})$ to the completed enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

We construct and prove in theorem 4.2.1 that there exists an homomorphism of Lie algebras

$$U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

which has two crucial properties. First it is possible, using only the structure of a vertex algebra, to build a complete topological associative algebra from $U(V)$ which we call $\widetilde{U}(V)$. In the case of $V = V_\kappa(\mathfrak{g})$ the algebra to homomorphism above induces a continuous homomorphism of algebras $\widetilde{U}(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ which we prove in theorem 4.2.2 to be an isomorphism. Therefore the vertex algebra $V_\kappa(\mathfrak{g})$ contains all the information of the enveloping algebra. Secondly we find that the composition

$$U(\zeta_\kappa(\mathfrak{g})) \to U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

has image contained in the center. Thus if the center of the vertex algebra is not trivial we can construct various central elements in the enveloping algebra. We prove a more precise statement: in corollary 4.6.2 we prove that under the assumption that point 2 of theorem 1.3.1 is true we are able to deduce that $\zeta(\mathfrak{g})$ topologically generates $Z_{\kappa_c}(\hat{\mathfrak{g}})$ (only at the critical level). This is done by considering various graded spaces associated to $Z_{\kappa_c}(\hat{\mathfrak{g}})$ and describe them as algebras functions on certain geometric spaces related to $\mathfrak{g}^*[[t]]$.

On the other hand it is shown in [Fre07] that the third part of theorem 1.3.1 implies the gometric description of the center of the enveloping algebra.

To sum up we reduced ourselves to treat two different (although related) problems: the first one is to show that $\operatorname{gr} \zeta(\mathfrak{g}) = \operatorname{Inv} \mathfrak{g}^*[[t]]$ and the second one is to identify $\zeta(\mathfrak{g})$ with the algebra of functions on the space of Opers. From this point onward we will forget about the enveloping algebra and restrict our attention to the vertex algebra setting.

Everything discussed so far is treated in chapter 4. We start with the construction of $U(V)$ as well as the definition of its Lie bracket, continue with the identification of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ with $\widetilde{U}(V_\kappa(\mathfrak{g}))$ and finally we study the center.

Invariants

We begin our quest to solve our two problems with facing the equality $\operatorname{gr}\zeta(\mathfrak{g}) = \operatorname{Inv}\ \mathfrak{g}^*[[t]]$. A good starting point is surely an efficient description of the space $\operatorname{Inv}\ \mathfrak{g}^*[[t]]$.

We start with the finite dimensional case. It is known that given a simple Lie algebra $\mathfrak{g}$ the space of invariants $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$ is a free polynomial algebra generated by certain homogeneous polynomials $P_1, \ldots, P_l$ with $l = \operatorname{rank}\mathfrak{g}$. It is reasonable to think that a description of $\mathbb{C}\left[\mathfrak{g}^*[[t]]\right]^{\mathfrak{g}[[t]]}$ may be deduced from this description of $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$.

This is indeed the case. We approach this problem using the formalism of **Jet Schemes**. To any scheme of finite type X over $\mathbb{C}$ we associate another scheme called JX, the $\mathbb{C}$ points of the latter are exactly the $\mathbb{C}[[t]]$ points of X. Thus to formally define $\mathfrak{g}^*[[t]]$ as a scheme we consider the Jet scheme $J\mathfrak{g}^*$. This formalism allows us to prove in theorem 4.5.2 that

$$\operatorname{Inv}\ \mathfrak{g}^*[[t]] = \mathbb{C}[P_{i,n}]_{i=1,\ldots,l;n<0}$$

where the $P_{i,n}$ are certain polynomials which are easily described from the polynomials P_i.

This description of $\operatorname{Inv}\ \mathfrak{g}^*[[t]]$ is still not enough. We move a little bit our problem to an analogous one which concerns a Verma module for $\hat{\mathfrak{g}}_{\kappa_c}$. The motivation behind this argument lies in the fact that this Verma module is easier to describe, as we will see in the next section.

Let $\hat{\mathfrak{n}}_+ := \mathfrak{n}_+ \oplus t\mathfrak{g}[[t]]$ and let $\widetilde{\mathfrak{b}}_+ := \mathfrak{b}_+ \oplus t\mathfrak{g}[[t]] = \mathfrak{h} \oplus \hat{\mathfrak{n}}_+$. Consider the trivial $\widetilde{\mathfrak{b}}_+ \oplus \mathbb{C}\mathbf{1}$ module $\mathbb{C}_0$ where $\widetilde{\mathfrak{b}}_+$ acts like 0 and $\mathbf{1}$ acts like the identity. Define the Verma module $\mathbb{M}_{0,\kappa_c}$ as the induced module

$$\mathbb{M}_{0,\kappa_c} := \operatorname{Ind}_{\widetilde{\mathfrak{b}}_+\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}_0$$

notice that this is a little bit larger than $V_{\kappa_c}(\mathfrak{g})$. Indeed there is a natural surjective $\hat{\mathfrak{g}}_{\kappa_c}$ linear morphism $\mathbb{M}_{0,\kappa_c} \to V_{\kappa_c}(\mathfrak{g})$. The problem of comparing the graded space of the $\mathfrak{g}[[t]]$ invariants on $V_{\kappa_c}(\mathfrak{g})$ with the space of $\mathfrak{g}[[t]]$ invariants of the graded space $\operatorname{gr} V_{\kappa_c}(\mathfrak{g})$ translates in this setting to the problem of comparing $\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}\right)$ with $\left(\operatorname{gr}\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}}_+}$. The common point of these two problems is the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}\right) & \longrightarrow & \operatorname{gr}\left(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}\right) \\
\big\downarrow & & \big\downarrow \\
\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}}_+} & \longrightarrow & \operatorname{gr}\left(V_{\kappa_c}(\mathfrak{g})\right)^{\mathfrak{g}[[t]]}
\end{array}$$

We are able to describe $\left(\operatorname{gr}\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}}_+}$ identifying $\operatorname{gr}\mathbb{M}_{0,\kappa_c}$ with the rings of functions on a certain geometric space, and a reasoning analogous to the one needed to compute $\operatorname{Inv}\ \mathfrak{g}^*[[t]]$ allows us to completely describe it and to show that the lower horizontal map is surjective.

Notice that if we could prove that the left vertical arrow is an isomorphism we could easily deduce that the right vertical arrow is an isomorphism as well. Thus we reduced ourselves to describe the space of invariants $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$.

Free field realization

A common tool to face both the problem of describing $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$ and the problem of giving a geometric interpretation of $\zeta(\mathfrak{g})$ is the so called **free field realization** of $V_{\kappa_c}(\mathfrak{g})$. It consists in an embedding of vertex algebras of $V_{\kappa_c}(\mathfrak{g})$ into a free field algebra which is a kind of vertex algebra particularly easy to study.

The idea behind this construction is found in the finite dimensional case: a way to construct the so called Harish-Chandra homomorphism

$$Z(\mathfrak{g}) \hookrightarrow \mathbb{C}[\mathfrak{h}^*]$$

which is essential to identify the center of the classical enveloping algebra $Z(\mathfrak{g})$ with the ring of functions on $\mathfrak{h}^*$ invariant by the action of the Weyl group, is the following.

Consider G to be the simply connected Lie group with $\mathfrak{g}$ as a Lie algebra and let N_+, H, N_- be the subgroups defined by the subalgebras $\mathfrak{n}_+, \mathfrak{h}, \mathfrak{n}_-$ respectively. Consider the left action of G on the quotient variety G/N_-. This action induces an action of $\mathfrak{g}$ by vector fields on any open subset of G/N_-, in particular on its open $B_+ = N_+ \times H$ orbit $U := B_+ \cdot [1] \simeq B_+$. This action induces an homomorphism of Lie algebras

$$\mathfrak{g} \to \mathrm{Vect}(N_+ \times H)$$

which actually factors through the Lie subalgebra $\mathrm{Vect}(N_+) \oplus \left(\mathbb{C}[N_+] \otimes \mathfrak{h}\right) \subset \mathrm{Vect}(N_+ \times H)$ where we identify $\mathfrak{h} \subset \mathrm{Vect}(H)$ with the constant right invariant vector fields. This homomorphism induces an homomorphism of associative algebras

$$U(\mathfrak{g}) \to D(N_+) \otimes \mathrm{Sym}\,\mathfrak{h} = D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*]$$

where $D(N_+)$ is the algebra of differential operators on N_+. This morphism turns out to have two remarkable properties: it is injective and it induces an homomorphism $Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$.

We construct the infinite dimensional analogue of this embedding and recast it in the language of vertex algebras. We describe in theorem 5.5.1 an homomorphism of vertex algebras

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$$

where $M_{\mathfrak{g}}$ should be understood as the vertex algebra analogue of an algebra of differential operators while π_0 is what is called an abelian vertex algebra and should be considered as the analogue of $\mathbb{C}[\mathfrak{h}^*]$ in this setting. In theorem 6.1.1 we prove that the above morphism is actually an embedding, considering the analogous statement for the finite dimensional case.

The whole construction is highly non trivial and presents various technical difficulties but on the other hand it highlights the role of the critical level in the study of affine algebras $\hat{\mathfrak{g}}_\kappa$.

We are going to see in what comes next how this construction helps us to move forward in the proof.

Wakimoto modules

First we are going to attack the description of $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$.

There is a natural notion of a module over a vertex algebra. We prove in theorem 6.3.1 that the notion of a $V_\kappa(\mathfrak{g})$ module and the notion of a $\hat{\mathfrak{g}}_\kappa$ smooth module coincide. In addition, given an homomorphism of vertex algebras $V \to W$ and a W module M, there is a canonical induced structure of V module on M obtained pullbacking the action of W. With this properties in mind we see that any $M_\mathfrak{g} \otimes \pi_0$ modules is automatically an $\hat{\mathfrak{g}}_{\kappa_c}$ module.

The algebras $M_\mathfrak{g}$ and π_0 are easy to describe. $M_\mathfrak{g}$ may be thought as generated by elements $a_{\alpha,n}, a^*_{\beta,m}$ with $\alpha, \beta \in \Phi_+$, $n, m \in \mathbb{Z}$ and commutation relations $[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{n,-m}$. On the other hand π_0 can be though as generated by elements $b_{i,n}$ with $i = 1, \dots, l$ and $n < 0$ which commute with each other. This description allows one to easily construct a lot of modules over these algebras.

We identify a wide class of $\hat{\mathfrak{g}}_{\kappa_c}$ modules, the so called **Wakimoto modules**. These are obtained considering a $M_\mathfrak{g} \otimes \pi_0$ modules of the form $L \otimes N$, where L is an $M_\mathfrak{g}$ module and L is a π_0 module, with the structure of $V_{\kappa_c}(\mathfrak{g})$ module and hence of a $\hat{\mathfrak{g}}_{\kappa_c}$ module. The free field realization is described by rather explicit formulas on the generators (see theorem 5.5.1), and therefore Wakimoto modules are not so difficult to describe.

Wakimoto modules come in the picture in the following way. In proposition 6.4.1 we establish an isomorphism of the Verma module $\mathbb{M}_{0,\kappa_c}$ with a certain Wakimoto module which we call W^+_{0,κ_c}. This identification, together with the explicit formulas typical of Wakimoto modules allow us to compute the space of invariants $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$.

In particular we notice that all the modules considered so far are actually modules for the extended affine algebra $\hat{\mathfrak{g}}'_{\kappa_c} := \hat{\mathfrak{g}}_{\kappa_c} \rtimes \mathbb{C}L_0$ where L_0 is the operator acting on $\hat{\mathfrak{g}}_{\kappa_c}$ as $-t\partial_t$. We are able to compute the character of the space of invariants $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$ with respect to the action of L_0, as well as the character of the invariants of the graded space $\left(\mathrm{gr}\,\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}}_+}$. We find that they are actually equal and therefore the inclusion

$$\mathrm{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}\right) \hookrightarrow \left(\mathrm{gr}\,\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}}_+}$$

is actually an isomorphism.

This allows us to conclude that $\mathrm{gr}\,\zeta(\mathfrak{g}) = \mathrm{Inv}\ \mathfrak{g}^*[[t]]$ as remarked before and therefore to describe the center of the completed enveloping algebra as the completion of a polynomial ring in an infinite number of variables. The definition of Wakimoto modules and the description of $\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}$ may be found in chapter 6.

Opers and Screening Operators

What remains to do is the identification of $\zeta(\mathfrak{g})$ with the algebra of functions on the space of Opers. The space of Opers is a classifying space for certain connections on the trivial ${}^LG$ bundle over the formal disc $D = \mathrm{Spec}\,\mathbb{C}[[t]]$. This space is described in the first part of chapter 8. After defining the space of Opers we immediately see that its algebra of functions is isomorphic to a free polynomial algebra in a countable number of variables

$$\mathbb{C}[\mathrm{Op}_{{}^LG}(D)] = \mathbb{C}[v_{i,n}]_{i=1,\dots,l;n<0}$$

Notice that **as commutative algebras** it is obvious, by what we have done so far, that $\zeta(\mathfrak{g})$ is isomorphic as a $\mathbb{C}$ algebra to the algebra of functions on $\mathrm{Op}_{^L G}(D)$. Our goal is to obtain a more canonical isomorphism, in particular we focus on constructing a $\mathrm{Der}\,\mathcal{O}$ equivariant isomorphism.

Recall that the group of automorphisms of the formal disc $\mathrm{Aut}\,\mathcal{O}$, and hence its Lie algebra $\mathrm{Der}\,\mathcal{O}$, naturally act on the Lie algebra $\hat{\mathfrak{g}}_\kappa$, for any value. This action induces a vertex algebra action of $\mathrm{Der}\,\mathcal{O}$ on $V_\kappa(\mathfrak{g})$ and hence a natural action on the center $\zeta_\kappa(\mathfrak{g})$. On the other hand the space of Opers is defined over the disc itself and therefore carries by its very definition an action of the group $\mathrm{Aut}\,\mathcal{O}$ and therefore its algebra of functions carries a natural $\mathrm{Der}\,\mathcal{O}$ action. The isomorphism we present is compatible with these actions. In the thesis we focus on the action of $\mathrm{Der}\,\mathcal{O}$ but it is possible to extend the result considering also the action of the group $\mathrm{Aut}\,\mathcal{O}$, whose action is anyway strictly connected to the action of $\mathrm{Der}\,\mathcal{O}$.

To proceed with the proof we introduce an auxiliary space: the space $\mathrm{MOp}_{^L G}(D)_{gen}$ of **generic Miura Opers** on the formal disc. It turns out that $\mathrm{MOp}_{^L G}(D)_{gen}$ fibers over $\mathrm{Op}_{^L G}(D)$ and it is actually an N_+ torsor over it. Under a specific trivialization $\mathrm{MOp}_{^L G}(D)_{gen} \simeq \mathrm{Op}_{^L G}(D) \times N_+$ we may write

$$\mathbb{C}[\mathrm{MOp}_{^L G}(D)_{gen}] = \mathbb{C}[\mathrm{Op}_{^L G}(D)] \otimes \mathbb{C}[N_+]$$

and

$$\mathbb{C}[\mathrm{Op}_{^L G}(D)] = \mathbb{C}[\mathrm{MOp}_{^L G}(D)]^{N_+} = \mathbb{C}[\mathrm{MOp}_{^L G}(D)]^{\mathfrak{n}_+}$$

A natural way to describe the algebra $\mathbb{C}[\mathrm{Op}_{^L G}(D)]$ in terms of the algebra $\mathbb{C}[\mathrm{MOp}_{^L G}(D)_{gen}]$ is to write down the infinitesimal action of the generators e_i of the Lie algebra $\mathfrak{n}_+$. This allows us to describe the algebra of functions on $\mathrm{Op}_{^L G}(D)$ as the intersection of the kernels of the operators e_i inside $\mathbb{C}[\mathrm{MOp}_{^L G}(D)_{gen}]$.

To link these construction with the vertex algebra setting we establish a $\mathrm{Der}\,\mathcal{O}$ equivariant isomorphism between

$$\pi_0 \simeq \mathbb{C}[\mathrm{MOp}_{^L G}(D)_{gen}]$$

We find that in a completely analogous way with respect to the finite dimensional case, the center $\zeta(\mathfrak{g})$ is mapped through the free field realization into π_0:

$$\zeta(\mathfrak{g}) \hookrightarrow \pi_0$$

Thus we embedded, in a $\mathrm{Der}\,\mathcal{O}$ equivariant way, the center $\zeta(\mathfrak{g})$ and the algebra $\mathbb{C}[\mathrm{Op}_{^L G}(D)]$ in the same ambient space. All we are left to do is to show that they are equal. This is done by considering another description of the operators e_i on $\mathbb{C}[\mathrm{MOp}_{^L G}(D)_{gen}]$.

In chapter 7 we introduce the so called **intertwining operators** $\overline{S}_i$ which are $\hat{\mathfrak{g}}_{\kappa_c}$ linear morphisms

$$\overline{S}_i : M_{\mathfrak{g}} \otimes \pi_0 \to \widetilde{W}^{(i)}_{0,0,\kappa_c}$$

where $\widetilde{W}^{(i)}_{0,0,\kappa_c}$ is a certain $\hat{\mathfrak{g}}_{\kappa_c}$ module described in the same chapter. We show that the subalgebra $V_{\kappa_c}(\mathfrak{g})$ lies in the intersection of the kernels of these operators. In particular the center $\zeta(\mathfrak{g})$ is contained in the intersection of the kernels of $\overline{V}_i : \pi_0 \to \pi_0 \subset \widetilde{W}^{(i)}_{0,0,\kappa_c}$ which are obtained by restricting $\overline{S}_i$ to π_0.

$$\zeta(\mathfrak{g}) \subset \bigcap_i \ker \overline{V}_i$$

Finally thanks to the explicit description we give of all these operators we find at the end of chapter 8 that the operators $\overline{V}_i$ coincide with the operators e_i under the isomorphism $\pi_0 \simeq \mathbb{C}[MOp_{^LG}(D)_{gen}]$. In particular we obtain an $\mathrm{Der}\,\mathcal{O}$ equivariant immersion

$$\zeta(\mathfrak{g}) \hookrightarrow \mathbb{C}[Op_{^LG}(D)]$$

To conclude the proof we use the equality $gr\,\zeta(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*[[t]]]^{\mathfrak{g}[[t]]}$ to compute the character of $\zeta(\mathfrak{g})$ under the action of $L_0 = -t\partial_t \in \mathrm{Der}\,\mathcal{O}$. We compute explicitly the character of $\mathbb{C}[Op_{^LG}(D)]$ and a comparison of the two characters shows that they are actually equal, so the above inclusion must be an isomorphism.

Chapter 2

Preliminaries

In this first chapter we give the basic definitions, notations and results used in the rest of the thesis. We focus on the theory of Lie algebras and on Algebraic Geometry.

2.1 Lie Algebras

Throughout the thesis the symbol $\mathfrak{g}$ will stand for a simple finite dimensional Lie algebra over $\mathbb{C}$. Given a fixed maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we consider the decomposition of $\mathfrak{g}$ in root spaces $\mathfrak{g} = \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, here $\Phi \subset \mathfrak{h}^*$ denotes the set of roots, these spaces are one dimensional. $l = \dim_{\mathbb{C}} \mathfrak{h}$ will be called the rank of $\mathfrak{g}$ and given a basis $\{\alpha_1, \ldots, \alpha_l\} = \Delta \subset \Phi$ we will denote by $e_i \in \mathfrak{g}_{\alpha_i}$ a fixed set of generators of the spaces $\mathfrak{g}_{\alpha_i}$.

There exists unique elements $f_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$ such that the triple e_i, h_i, f_i is an $\mathfrak{sl}_2$-triple, h_i does not depend by the choice of e_i but f_i does. We will always The numbers $a_{ij} = \alpha_j(h_i)$ are called the Cartan integers. The matrix (a_{ij}) is called the Cartan matrix, it is positive definite and characterizes $\mathfrak{g}$. Moreover given a symmetrizable positive definite matrix one can reconstruct a Lie algebra associated to it.

The elements e_i, h_i, f_i are a set of generators and $\mathfrak{g}$ is isomorphic to the free Lie algebra generated by them subjected to the following Serre Relations:

- (S1) $[h_i, h_j] = 0$
- (S2) $[e_i, f_j] = h_i \delta_{ij}$
- (S3) $[h_i, e_j] = a_{ij} e_j$ and $[h_i, f_j] = -a_{ij} f_j$
- (S^+_{ij}) $ad(e_i)^{-a_{ij}+1}(e_j) = 0$
- (S^+_{ij}) $ad(f_i)^{-a_{ji}+1}(f_j) = 0$

This explicit description allows us to define an involution ι for such algebras: $\iota(e_i) = f_i, \iota(h_i) = -h_i, \iota(f_i) = e_i$.

We call ${}^L\mathfrak{g}$ the Langlands dual Lie algebra it is defined as the algebra given by the transposed Cartan matrix i.e. ${}^L a_{ij} = a_{ji}$. Some further notation:

- We will denote by $\omega_i^\vee$ the element of $\mathfrak{h}$ defined by the equations $\alpha_j(\omega_i^\vee) = \delta_{ij}$;
- We will denote by $\rho^\vee$ the element of $\mathfrak{h}$ defined by the equations $\alpha_i(\rho^\vee) = 1\ \forall i = 1, \dots, l$, we naturally have $\rho^\vee = \sum_i \omega_i^\vee$.

2.1.1 Principal Gradation, exponents

We states some results concerning the exponents of a simple Lie algebra $\mathfrak{g}$ and their relation to the ring of invariants $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$ we refer to [Kos63].

Consider the element $p_{-1} \in \mathfrak{g}$ defined to be the sum of the generators for the lower nilpotent subalgebra $\mathfrak{n}_-$: $p_{-1} = \sum_i f_i$.

Remark 2.1.1. There exists coefficients m_i such that $(\sum_i m_i e_i, 2\rho^\vee, p_{-1})$ is an $\mathfrak{sl}_2$ triple. Indeed for every choice of m_i we have that $[2\rho^\vee, \sum_i m_i e_i] = 2\sum_i m_i e_i$ so we just have to choose them in order to have $2\rho^\vee = [\sum_i m_i e_i, p_{-1}] = \sum_i m_i h_i$. This is possible since h_i is a basis of $\mathfrak{h}$.

Definition 2.1.1. The decomposition

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$$

induced by the adjoint action of $\rho^\vee$ is called the **principal gradation** of $\mathfrak{g}$. Notice that $\mathfrak{g}_i$ is exactly the direct sum of root spaces for roots of height i. In particular

$$\mathfrak{b} = \bigoplus_{i \geq 0} \mathfrak{g}_i = \bigoplus_{i \geq 0} \mathfrak{b}_i$$

The morphism $ad(p_{-1}) : \mathfrak{b}_{i+1} \to \mathfrak{b}_i$ is injective, and therefore we may find a subspace $V_i \subset \mathfrak{b}_i$ such that

$$\mathfrak{b}_i = ad(p_{-1})\mathfrak{b}_{i+1} \oplus V_i$$

Definition 2.1.2. The natural numbers i such that $V_i \neq 0$ are called **exponents** of $\mathfrak{g}$, and $\dim V_i$ is called the **multiplicity** of the exponent i. Note that $V_0 = 0$ and define

$$V = \bigoplus_{i \geq 0} V_i \subset \mathfrak{n}$$

In addition we define

$$E := \{d_1, \dots, d_l\}$$

the set of exponents counted with multiplicity.

Finally we state the theorem linking the exponents of $\mathfrak{g}$ to the algebra of invariants.

Theorem 2.1.1 (Kostant)**.** *The ring of invariants*

$$\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$$

is freely generated by homogeneous polynomials $P_i : i = 1, \dots, l$. *The degree of* P_i *is exactly* $d_i + 1$.

2.1.2 Affine Kac-Moody algebras

We extend the definition of the affine Kac-Moody algebra in the non-simple case. Consider any Lie algebra $\mathfrak{g}$ and an invariant inner product κ on it. Then the following formula defines a cocycle in $H^2(L\mathfrak{g}, \mathbb{C})$

$$c(X \otimes f, Y \otimes g) = \kappa(X, Y)\mathrm{Res}_{t=0} f dg$$

the associated central extension in called the **affine algebra** $\hat{\mathfrak{g}}_\kappa$. When $\mathfrak{g}$ is simple it will be called affine Kac-Moody algebra.

Representations of affine Kac-Moody algebras

We will often consider representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$ which are modules for the extended algebra

$$\hat{\mathfrak{g}}'_\kappa := \hat{\mathfrak{g}}_\kappa \rtimes L_0$$

where L_0 is the grading operator acting on $\hat{\mathfrak{g}}_\kappa$ by $L_0 = -t\partial_t$. We require additionally that in these representations the action of $\mathfrak{h} \oplus \mathbb{C}L_0$ is semisimple with finite dimensional eigenspaces, the latter will be also called weight spaces. We will denote weights by couples (λ, μ) where $\lambda : \mathfrak{h} \to \mathbb{C}$ is a linear functional and $\mu \in \mathbb{C}$ is the value of the eigenvalue relative to $-L_0 = t\partial_t$.

Note that $\hat{\mathfrak{g}}_\kappa$ with the adjoint representation of $\hat{\mathfrak{g}}'_\kappa$ is a module with the above properties (actually one should consider the subalgebra $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$) its weight spaces are of the form

$$(\hat{\mathfrak{g}}_\kappa)_{(\alpha,n)} = \mathfrak{g}_\alpha \otimes t^n$$

We define the set of positive roots of $\hat{\mathfrak{g}}_\kappa$ as follows

$$\hat{\Phi}_+ = \{(\alpha, n) : \alpha \in \Phi_+, n \geq 0\} \cup \{(-\alpha, n) : \alpha \in \Phi_+, n > 0\} \cup \{(0, n) : n > 0\}$$

The subspace relative to the positive roots is the Lie subalgebra $\hat{\mathfrak{n}}_+ := \mathfrak{n}_+ \oplus t\mathfrak{g}[[t]]$.

2.2 Algebraic Geometry

As algebraic geometry is concerned we will work in the category $\mathbf{Sch}_\mathbb{C}$ of Schemes over $\mathbb{C}$, we will give in this section an overview of all the language and the tools we will need in the thesis.

2.2.1 Schemes as functors

The natural embedding $\mathbf{Alg}_\mathbb{C}^{op} \to \mathbf{Sch}_\mathbb{C}$ is fully faithful, its essential image will be called the category of affine schemes. Using the fact that every scheme admits an open cover of affine subschemes one can prove a stronger version of the Yoneda Lemma:

Proposition 2.2.1. *The restriction of the Yoneda Embedding $\mathbf{Sch}_\mathbb{C} \to \mathrm{Fun}(\mathbf{Sch}_\mathbb{C}^{op}, \mathbf{Set})$ to $\mathrm{Fun}(\mathbf{Alg}_\mathbb{C}, \mathbf{Set})$*

$$\mathbf{Sch}_\mathbb{C} \to \mathrm{Fun}(\mathbf{Alg}_\mathbb{C}, \mathbf{Set}) \qquad X \mapsto (R \mapsto X(R) = \mathrm{Hom}_\mathbb{C}(\mathrm{Spec}\, R, X))$$

is still fully faithful.

The above proposition is true in a more general setting, making the substitution $\mathbb{C} \mapsto R$ for a $\mathbb{C}$-algebra R. We will call a functor $F : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$ *representable* if it is naturally isomorphic to a scheme $X \in \mathbf{Sch}_{\mathbb{C}}$. In virtue of the above proposition we will make no difference between a scheme X and its functor of points $R \mapsto X(R)$. We may also talk of functions in terms of functors.

Definition 2.2.1. Let X be a functor $X : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$. Define

$$\mathbb{C}[X] := \left\{ f : X \to \mathbb{A}^1 \right\}$$

the set of natural transformations from X to the scheme $\mathbb{A}^1$. The multiplication and the addition morphisms $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ give the structure of a $\mathbb{C}$ algebra to $\mathbb{C}[X]$. If X is a scheme then $\mathbb{C}[X]$ is obviously equal to the ring of regular functions on X.

If the scheme X is affine (i.e. isomorphic to the spectrum of a $\mathbb{C}$ algebra) the Yoneda Lemma provides us with a universal Element $\xi_X \in F(X)$. Here we are identifying with a slight abuse of notation X with its ring of functions, the difference will be clear from the context. In this case the isomorphism $\varphi : X \to F$ is given by the following: if $f \in X(R) = \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{C}}}(X, R)$ then $\varphi_R(f) = F(f)(\xi_X)$ where $F(f) : F(X) \to F(R)$ is the morphism associated to f by functoriality.

We list a couple of useful remarks:

- The category $\mathbf{Sch}_{\mathbb{C}}$ admits arbitrary fibered products. Given three schemes X, Y, Z and maps $Y \to X, Z \to X$ the functor of points associated to $Y \times_X Z$ equals the functor $R \mapsto Y(R) \times_{X(R)} Z(R)$;

Sheaves for the flat topology

Representable functors have various nice property, the one we are interested in is the property of being a sheaf for the flat topology. We follow [Mil17][Definition 5.65]

Definition 2.2.2. A functor $F : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$ is said to be a *sheaf for the flat topology* if the following axioms hold:

- (local) For all $\mathbb{C}$-algebras R_i for $i = 1, \dots, n$ the map $F(R_1 \times \cdots \times R_n) \to F(R_1) \times \cdots \times F(R_n)$ is an isomorphism;
- (descent) For any faithfully flat map $R \to R'$ the sequence

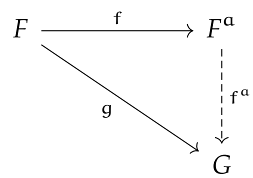
$$F(R) \to F(R') \rightrightarrows F(R' \otimes_R R')$$

is exact, i.e., the first map is the equalizer of the second couple of maps.

One can check that representable functors (coming from schemes) are sheaves for the flat topology. Therefore when we are going to try to define geometric spaces (i.e. schemes) through functors we are interested in functors that are sheaves for the flat topology. Fortunately there is a way to associate to a functor a canonical flat sheaf.

Theorem 2.2.1. *Let F be a functor $F : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$. Then there exists a couple (F^a, f) such that F^a is a sheaf for the flat topology and $f : F \to F^a$ is a morphism which satisfy the following universal property:*

for every other flat sheaf G and every morphism $g : F \to G$ there exists a unique morphism $f^a : F^a \to G$ such that the following diagram is commutative:



Such a universal object must of course be unique up to unique isomorphism.

Definition 2.2.3. A subfunctor F of a functor G is said to be a **fat subfunctor** if for every element $x \in G(R)$ there exists a finite faithfully flat family R_i (i.e. some R_i with a faithfully flat morphism $R \to \Pi_i R_i$) such that the image of x in $G(R_i)$ belongs to $F(R_i)$ for each i. This is equivalent to asking that G is the sheafification of F.

Proposition 2.2.2. *Let F be a fat subfunctor of G. Then the canonical map $F^a \to G$ is an isomorphism, in particular for any flat sheaf H*

$$\mathrm{Hom}(F, H) = \mathrm{Hom}(G, H)$$

2.2.2 Tangent space, vector fields

We define here the notions of the tangent space.

Definition 2.2.4. Let X be a $\mathbb{C}$ scheme (or any functor $X : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Set}$). Let TX be the functor of $\mathbb{C}$-algebras defined by

$$TX(R) := X(R[\epsilon])$$

If X is a scheme of finite type TX is representable by a scheme of finite type over $\mathbb{C}$. The natural maps $R \to R[\epsilon]$ and $(\epsilon = 0) : R[\epsilon] \to R$ define morphisms

$$0 : X \to TX \qquad \text{and} \qquad \pi : TX \to X$$

whose composition is the identity on X.

As an example consider $X = \mathrm{Spec}\,\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ then

$$TX(R) = X(R[\epsilon]) = \left\{(r_1 + \epsilon r_1^\epsilon, \dots, r_n \epsilon r_n^\epsilon) : f_i(r + \epsilon r^\epsilon) = f_i(r) + \epsilon \langle r^\epsilon, \nabla f_i(r) \rangle = 0 \,\forall i \right\}$$

The functor TX results therefore representable by the affine scheme

$$\mathrm{Spec}\,\mathbb{C}[x_i, x_i^\epsilon]/(f_j(x), \langle x^\epsilon, f_j(x) \rangle)$$

so TX effectively corresponds with the intuitive notion of tangent space.

Definition 2.2.5. Given a $\mathbb{C}$ scheme X (or a functor of $\mathbb{C}$ algebras) a **vector field** on X is a morphism

$$\nu : X \to TX \quad \text{such that } \pi \circ \nu = \mathrm{id}_X$$

The set of vector fields form naturally a $\mathbb{C}$ vector space. If X is affine of finite type one may check that

$$\mathrm{Vect}(X) = \mathrm{Der}\,\mathbb{C}[X]$$

Therefore for a general scheme X of finite type the sheaf of $\mathbb{C}$-vector spaces $U \mapsto \mathrm{Vect}(U)$ has a natural local structure of a Lie algebra which induces a structure Lie algebra on the sheaf Vect. This sheaf naturally acts on the sheaf of sections (through the action of $\mathrm{Der}\,\mathbb{C}[U]$).

One may intrinsically define a 'bracket' product on $\mathrm{Vect}(X)$ for any functor X, which if X is sufficiently well behaved gives to $\mathrm{Vect}(X)$ the structure of a Lie algebra. We will omit this general construction on limit ourselves to describe the Lie algebra of $\mathrm{Vect}(X)$ on the specific situations we will encounter.

Anyway we may define an action of $\mathrm{Vect}(X)$ on the algebra $\mathbb{C}[X]$ as follows. Consider a function f and a vector field ν define $\nu \cdot f$ as the composition

$$X \xrightarrow{\nu} TX \xrightarrow{Tf} T\mathbb{A}^1 \xrightarrow{\frac{d}{d\epsilon}} \mathbb{A}^1$$

where $\frac{d}{d\epsilon}$ is the morphism defined on R points by $T\mathbb{A}^1(R) \ni r + \epsilon r^\epsilon \mapsto r^\epsilon \in \mathbb{A}^1(R)$. This definitions coincides with the ones we gave for the finite type case.

2.2.3 Group functors and Group schemes

We are going to give here the notions and tools we will use regarding algebraic group schemes. We will focus on group schemes defined over $\mathbb{C}$.

Definition 2.2.6. A **group functor** over $\mathbb{C}$ is a functor

$$G : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Grp}$$

An homomorphism of group functors is simply a natural transformation, which automatically has to preserve the group structure.

If the underlying functor to **Set** is representable by a scheme G is called a **group scheme**. An homomorphism of group schemes is an homomorphism of group functors, by the Yoneda lemma it induces a morphism of the corresponding represented schemes.

All groups schemes we are interested in will be affine and algebraic (i.e. of finite type over $\mathbb{C}$), we will refer to them simply as algebraic groups. They are easily seen to be smooth.

From time to time we will encounter group schemes over $\mathbb{C}$ which are not of finite type, all such groups will be anyway closely related to algebraic groups.

Definition 2.2.7. The **Lie algebra** of a group functor is defined to be the functor of R-algebras

$$\mathfrak{g}(R) := \mathrm{Lie}(G)(R) := \ker(G(R[\epsilon]) \to G(R))$$

It is possible to define functorial maps

$$\mathfrak{g}(R) \times \mathfrak{g}(R) \xrightarrow{\quad + \quad} \mathfrak{g}(R)$$

$$\{*\} \xrightarrow{\quad 0 \quad} \mathfrak{g}(R)$$

$$R \times \mathfrak{g}(R) \xrightarrow{\qquad} \mathfrak{g}(R)$$

Which make $\mathfrak{g}(R)$ into an R module. If the group is algebraic $\mathfrak{g}(\mathbb{C})$ is a finite dimensional $\mathbb{C}$ vector space and there is a natural isomorphism

$$\mathfrak{g}(R) = \mathfrak{g}(\mathbb{C}) \otimes_{\mathbb{C}} R$$

So $\mathfrak{g}$ in this case is a vector space scheme.

Remark 2.2.1. To a morphism of group functors $f : G \to H$ is possible to associate a morphism between the corresponding Lie algebras

$$\mathrm{Lie}(f)(R)\mathfrak{g}(R) \to \mathfrak{h}(R) \qquad \mathrm{Lie}(f)(R) = f(R[\epsilon])_{|\mathfrak{g}(R)}$$

This association is of course functorial.

Example 2.2.1 (vector spaces)**.** Let V be a vector space over $\mathbb{C}$. Consider the functor of $\mathbb{C}$ algebras

$$V_a(R) = V \otimes_{\mathbb{C}} R \qquad \mathrm{id} \otimes f =: V_a(f) : V_a(R_1) \to V_a(R_2)$$

where $f : R_1 \to R_2$ is a morphism of $\mathbb{C}$ algebras.

The choice of a basis $(v_i)_{i \in I}$ for V result in a functorial isomorphism $V(R) = R^I$ so V_a is actually representable by a scheme. The functorial morphisms

$$V_a(R) \times V_a(R) \xrightarrow{\quad + \quad} V_a(R)$$

$$\{*\} \xrightarrow{\quad 0 \quad} V_a(R)$$

$$R \times V_a(R) \xrightarrow{\qquad} V_a(R)$$

defining the structure of R module on $V(R)$ induce morphisms of schemes $V_a \times V_a \to V_a$, $\mathbb{C} \to V_a$ and $\mathbb{A}^1 \times V_a \to V_a$. A scheme equipped with those map, with the appropriate commutative diagrams will be called a vector space scheme

Example 2.2.2 ($GL(V)$)**.** Let V be a complex vector space. Define $GL(V)$ as the group functor

$$GL(V)(R) := GL_R(V_R)$$

where as in the previous example $V_R := V \otimes_{\mathbb{C}} R$ for a $\mathbb{C}$ algebra R. Choosing a basis for V gives the usual isomorphism of $GL(V)$ with an open subscheme of $\mathbb{A}^{n^2}$ therefore $GL(V)$ is actually an algebraic group.

It's easy to see that the automorphisms $A \in GL(V)(\mathbb{C}[\epsilon])$ which are the identity at $\epsilon = 0$ are exactly those of the form

$$\mathrm{id} + \epsilon\varphi \qquad \varphi \in \mathrm{End}\, V$$

for any $\varphi \in \mathrm{End}\, V$. With this expression we mean the automorphism of $V \otimes \mathbb{C}[\epsilon]$ given by

$$v + \epsilon w \mapsto v + \epsilon w + \epsilon\big(\varphi(v) + \epsilon\varphi(w)\big)$$

This proves that $\mathrm{Lie}(GL(V))(\mathbb{C}) = \mathrm{End}\, V$, and it is not difficult to extend this result to $\mathrm{Lie}(GL(V))(R) = \mathrm{End}_R V_R$.

Definition 2.2.8. By an **action** of a group functor (or a group scheme) G on a functor (or a scheme) X we mean a map $\mu : G \times X \to X$ such that the following diagrams are commutative:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}} & G \times X \\ \downarrow{\scriptstyle \mathrm{id}\times\mu} & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow[\mu]{} & X \end{array} \qquad\qquad \begin{array}{ccc} * \times X & \xrightarrow{e \times \mathrm{id}} & G \times X \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \mu} \\ & & X \end{array}$$

where $m : G \times G \to G$ is the multiplication map induced by the group structure.

Notice that given a $\mathbb{C}$ point $g : * \to G$ the composition

$$X \xrightarrow{=} * \times X \xrightarrow{g \times \mathrm{id}} G \times X \xrightarrow{\mu} X$$

is an automorphism of X with inverse $g^{-1} \in G(\mathbb{C})$.

Associated to a group functor acting a space on there are several classical actions. Suppose we are given an action $G \times X \to X$

- (Action of $G(\mathbb{C})$ on $\mathbb{C}[X]$) For $g \in G(\mathbb{C})$ and $f \in \mathbb{C}[X]$ define

  $$g \cdot f := f \circ g^{-1}$$

  where g^{-1} is the automorphism of X induced by g^{-1}. Notice that on R points $g \cdot f$ is read as follows

  $$(g \cdot f)(x) = f(g^{-1}x) \quad \text{where } x \in X(R) \text{ and } g^{-1} \in G(\mathbb{C}) \to G(R)$$

  In particular this action of $G(\mathbb{C})$ on $\mathbb{C}[X]$ preserves the $\mathbb{C}$ algebra structure.

- (Action of $\mathfrak{g}(\mathbb{C})$ on $\mathbb{C}[X]$) Let $\xi \in \mathfrak{g}(\mathbb{C})$ and $f \in \mathbb{C}[X]$ consider the composition,

  $$\mathfrak{g} \times X \longrightarrow T(G \times X) \xrightarrow{T\mu} TX \xrightarrow{Tf} T\mathbb{A}^1 \xrightarrow{\frac{d}{d\epsilon}} \mathbb{A}^1$$

  where the first map is given the inclusion $\mathfrak{g}(R) \times X(R) \subset G(R[\epsilon]) \times X(R[\epsilon])$, the second and the first map are the $T\mu$ and Tf respectively while the last map is '$\frac{d}{d\epsilon}$' which on R points is defined as $r + \epsilon r_\epsilon \mapsto r_\epsilon$.

Define $\xi \cdot f$ as the composition

$$X \xrightarrow{=} * \times X \xrightarrow{-\xi \times id} \mathfrak{g} \times X \longrightarrow T(G \times X) \xrightarrow{T\mu} TX \xrightarrow{Tf} T\mathbb{A}^1 \xrightarrow{\frac{d}{d\epsilon}} \mathbb{A}^1$$

This induces a linear action of $\mathfrak{g}(\mathbb{C})$ o $\mathbb{C}[X]$, which actually consists on derivations, we will call such actions 'actions by vector fields'. If X is a scheme and U is an open subscheme of X the above diagram is actually well defined under the substitution $X \mapsto U$ this is essentially due to the fact that if $X \in U(R)$ and $\xi \in \mathfrak{g}(R)$ $\xi \cdot x \in U(R[\epsilon])$. Therefore given an action of G on X it is defined an action of $\mathfrak{g}(\mathbb{C})$ by derivation on $\mathbb{C}[U]$ for every open subscheme U of X.

Definition 2.2.9. A **representation** of an algebraic group on a vector space V is an homomorphism of group functors

$$G \to GL(V)$$

A representation naturally induces an action of G on V_a. Such actions are called linear.

Remark 2.2.2. By functoriality of Lie to every representation of G on a vector space V we obtain a linear homomorphism

$$Lie(G) \to End\, V$$

In particular if G is algebraic, we may consider the adjoint representation of G on its Lie algebra $\mathfrak{g}$ (note that if $g \in G(R)$ and $\xi \in \mathfrak{g}(R) \subset G(R[\epsilon])$

$$Ad : G \to GL(\mathfrak{g}) \qquad g \mapsto (\xi \mapsto g\xi g^{-1})$$

which may be checked to be linear. By functoriality of Lie It induces a map

$$ad : \mathfrak{g} \to End\, \mathfrak{g}$$

one may also check that $[x, y] := ad(x)(y)$ defines a Lie bracket on $\mathfrak{g}$, this is the only definition of a bracket which is functorial (i.e. for every representation V the map $\mathfrak{g} \to End\, V$ is a Lie algebra homomorphism), see [Mil17].

2.2.4 Fixed points, quotients

Consider a given action of an group scheme G on a scheme X: $\mu : G \times X \to X$. We introduce some spaces of invariant functions.

Definition 2.2.10. Let's group a list of definitions

- ($\mathbb{C}[X]^G$) We say that $f \in \mathbb{C}[X]$ is G invariant if the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & \mathbb{A}^1 \end{array}$$

  Where π_2 is the projection on the second factor. Let $\mathbb{C}[X]^G$ the set of G invariant functions. It is a subalgebra of $\mathbb{C}[G]$.

- ($\mathbb{C}[X]^{G(\mathbb{C})}$) Define $\mathbb{C}[X]^{G(\mathbb{C})}$ to be the set of invariant functions for the linear action of $G(\mathbb{C})$ on $\mathbb{C}[X]$
- ($\mathbb{C}[X]^{\mathfrak{g}}$) Let $f \in \mathbb{C}[X]$ be a function. We say that f is $\mathfrak{g}$ invariant if the composition

$$\mathfrak{g} \times X \longrightarrow T(G \times X) \xrightarrow{T\mu} TX \xrightarrow{Tf} T\mathbb{A}^1 \xrightarrow{\frac{d}{d\epsilon}} \mathbb{A}^1$$

  is identically 0;
- ($\mathbb{C}[X]^{\mathfrak{g}(\mathbb{C})}$) Define the ring $\mathbb{C}[X]^{\mathfrak{g}(\mathbb{C})}$ as the ring of invariant function for the action of $\mathfrak{g}(\mathbb{C})$ on $\mathbb{C}[X]$. Since $\mathfrak{g}(\mathbb{C})$ acts by derivations it is a subalgebra of $\mathbb{C}[X]$

These spaces are all closely related.

Proposition 2.2.3. *The following inclusions hold for any group scheme G and for every scheme X:*

$$\mathbb{C}[X]^G \subset \mathbb{C}[X]^{G(\mathbb{C})}$$
$$\mathbb{C}[X]^{\mathfrak{g}} \subset \mathbb{C}[X]^{\mathfrak{g}(\mathbb{C})}$$
$$\mathbb{C}[X]^G \subset \mathbb{C}[X]^{\mathfrak{g}}$$

If in addition G and X are of finite type and G is connected , these inclusions are all equalities.

Proof. All the above inclusions are obvious. So assume that G and X are of finite type. Then equality of morphisms may be checked on $\mathbb{C}$ points, which is exactly equivalent to say that the first two inclusions are equalities.

Finally the equality $\mathbb{C}[X]^{G(\mathbb{C})} = \mathbb{C}[X]^{\mathfrak{g}(\mathbb{C})}$ is a classical result of representation theory, which actually holds for any representation: $V^{G(\mathbb{C})} = V^{\mathfrak{g}(\mathbb{C})}$. □

When X is an affine schemes invariant functions are closely related to the notion of quotients. We stick to this setting, let's $\mu : G \times X \to X$ be an action of an algebraic group on an algebraic scheme X. Given any scheme Y and a morphism $f : X \to Y$ we say that f is G invariant if the following diagram is commutative:

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Definition 2.2.11. Let p be a G invariant map $p : X \to Y$. We say that the couple (Y, p) is a **categorical quotient** of X for the action of G if it is universal among all G invariant maps. By universal we mean that give any other G-invariant map $f : X \to Z$ there exists a unique morphism $F : Y \to Z$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} \\ & & Z \end{array}$$

There is another notion of quotient, which involves the notion of invariant functions.

Definition 2.2.12. Let p be a G invariant map $p : X \to Y$. We say that the couple (Y, p) is **geometric quotient** if the following properties are satisfied:

- p is surjective (on $\mathbb{C}$ points) and the natural map

$$G \times X \to X \times_Y X$$

  is surjective (on $\mathbb{C}$ points). Equivalently the $\mathbb{C}$ fiber of p consists of non-empty G orbits;

- p is submersive, which is to say that $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X;

- The map of sheaves $p^{\#} : \mathcal{O}_Y \to p_*\mathcal{O}_X$ is injective and its image is exactly the sheaf of G invariant functions. That is to say $\mathcal{O}_Y(U) = \mathcal{O}_X(p^{-1}(U))^G$ (note that G actually acts on every open subset of the form $p^{-1}(U)$).

Given an arbitrary action of G on X both the categorical and the geometric quotient must not necessarily exist.

The following proposition will be very useful for us and give a couple of criterions for the existence of quotients. Here both G and X are assumed to be algebraic over $\mathbb{C}$ (i.e. of finite type).

Proposition 2.2.4. *If the geometric quotient exists it is also a categorical quotient.*

The following may be found in [MFK94][Proposition 0.2]

Theorem 2.2.2. *Suppose X and Y are normal irreducible noetherian $\mathbb{C}$ schemes of finite type and let $p : X \to Y$ be a G invariant dominant morphism. Suppose in addition that the maps*

$$p : X \to Y \qquad G \times X \to X \times_Y X$$

are surjective (on $\mathbb{C}$ points). Then the couple (Y, p) is a geometric quotient for the action of G on X.

2.2.5 The formal disc

We give here an overview of what we mean by formal disc and all the constructions relative to it.

Definition 2.2.13. We call $D = \mathrm{Spec}\, \mathbb{C}[[t]]$ the formal disc over $\mathbb{C}$, $D_n = \mathrm{Spec}\, (\mathbb{C}[t]/(t^n))$ the n-th formal disc over $\mathbb{C}$

We have natural closed immersions $D_n \to D_m$ for $m \geq n$ and $D_n \to D$. These immersions form a cone over D. Actually it turns D is the colimit (in the category of functors $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{C}}, \mathbf{Set})$, this is simply because $C[[t]]$ is the projective limit in the category of $\mathbb{C}$-algebras of the rings $\mathbb{C}[t]/(t^n)$. Since it is a scheme and the category $\mathbf{Sch}_{\mathbb{C}}$ is a full subcategory of $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{C}}, \mathbf{Set})$ it turns out that D is also the colimit of D_n in the category $\mathbf{Sch}_{\mathbb{C}}$.

The same definitions work over an arbitrary $\mathbb{C}$-algebra R.

Definition 2.2.14. We call $D_R = \mathrm{Spec}\, R[[t]]$ the formal disc over R and $(D_n)_R = \mathrm{Spec}\, R[t]/(t^n)$ the n-th formal disc over R. As in the case $R = \mathbb{C}$, D_R is the colimit of $(D_n)_R$ in the category $\mathrm{Fun}(\mathbf{Alg}_R, \mathbf{Set})$ and hence it is also the colimit of $(D_n)_R$ in the category of R-schemes

In virtue of the interpretation of D and D_R as a limit of the n-th formal discs, which, being finitely generated R-algebras, satisfy nicer properties, we will be interested in 'continuous' objects on D, obtained from a limit of objects on D_n. This will be made more precise every time we introduce a new object.

We call the '0 point' of D_R as the R point defined by the morphism of R algebras $ev_0 : R[[t]] \to R$ which sends $t \mapsto 0$. Given a map of schemes $\varphi : D_R \to X$ of R-schemes we will call $\varphi(0) : \operatorname{Spec} R \to X$ the map obtained by restricting φ to the 0 point $\operatorname{Spec} R \to D_R$.

The following lemma will be very useful.

Lemma 2.2.1. *Let X be an R-scheme and U an open subscheme of X. Then there is a natural correspondence*

$$U(R[[t]]) \longleftrightarrow \{\varphi \in X(R[[t]]) : \varphi(0) \in U(R)\}$$

Proof. Since morphism of R-schemes $\varphi : \operatorname{Spec} R[[t]] \to U$ are in natural correspondence with morphisms $\varphi : \operatorname{Spec} R[[t]] \to X$ whose image is contained in U we have to show that the image of a morphism $\varphi : \operatorname{Spec} R[[t]] \to X$ is contained in U if and only if the image of $\varphi(0)$ is contained in U. The 'only if' part is obvious.

So consider a morphism $\varphi : \operatorname{Spec} R[[t]] \to X$ such that $\varphi(0)$ has image in U. We can restrict ourselves to check that the closed points are mapped in U since if any point is mapped in $X \setminus U$ then its closure is mapped in $X \setminus U$. Let $p \in \operatorname{Spec} R[[t]]$ be a closed point. If $t \in p$ then p corresponds to a closed point of $\operatorname{Spec} R$ so its mapped in U by hypothesis. Suppose by contradiction that there is a maximal ideal p such that t is not in p then by maximality there exists $r(t) \in R[[t]]$ and $x \in p$ such that $1 = x + r(t)t$ so $x = 1 - r(t)t$ but this is an invertible element of $R[[t]]$ hence the contradiction. □

We introduce now the continuous cotangent module over the formal disc D_R over R. Consider first the case of the n-th discs, it is easy to check that

$$\Omega^1_{(D_n)_R/R} = R[t]/(t^{n-1})dt$$

this is also a module over $R[[t]]$ and it is natural to define the continuous cotangent module over $R[[t]]$ as follows

Definition 2.2.15. The modules $\Omega^1_{(D_n)_R/R}$ form a projective (???) system of $R[[t]]$ modules. We define

$$\Omega^{1,cont}_{D_R/R} = \varprojlim \Omega^1_{(D_n)_R/R}$$

This definition is independent of a choice of a coordinate t. And given an isomorphism $D_R = \operatorname{Spec} R[[t]]$ we have a natural isomorphism

$$\Omega^{1,cont}_{D_R/R} = R[[t]]dt$$

therefore $\Omega^{1,cont}_{D_R/R}$ is a free $R[[t]]$-module of rank 1, for instance it is self dual.

The group $\operatorname{Aut} \mathcal{O}$

We will describe in this section the group functor $\operatorname{Aut} \mathcal{O}$. We will see that considering it as a functor will give us some useful insights on its Lie algebra that do not appear if we naively consider only its $\mathbb{C}$ points.

Definition 2.2.16. Let $\mathrm{Aut}\,\mathcal{O}$ be the group functor defined from $\mathbf{Alg}_{\mathbb{C}}$ to $\mathbf{Grp}$ as

$$\mathrm{Aut}\,\mathcal{O} : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Grp} \qquad \mathrm{Aut}\,\mathcal{O}(R) := \mathrm{Aut}_{\,R,cont}(R[[t]])$$

Where by a continuous automorphism we mean an automorphism which is continuous for the topology generated by the ideals (t^n).

Remark 2.2.3. Every such automorphism is determined by its image $\rho(t)$. It is straightforward that $\rho(t) = \sum_{n\geq 0} r_i t^i$ defines an automorphism if and only if r_0 is nilpotent and r_1 is invertible.

We have therefore a natural isomorphism

$$\mathrm{Aut}\,\mathcal{O}(R) = \big\{\rho(t) = \sum_{n\geq 0} r_i t^i \,:\, r_0 \text{ nilpotent and } r_1 \text{ invertible}\big\}$$

Note that the group structure under this isomorphism is read

$$\sigma \cdot \rho(t) = \rho(\sigma(t))$$

Because of the nilpotency condition the group $\mathrm{Aut}\,\mathcal{O}$ is not representable by a scheme, but it is actually an ind-scheme (i.e. a direct limit of schemes in the category of functors). Indeed the subgroups

$$\mathrm{Aut}\,\mathcal{O}_n(R) := \big\{\rho(t) = \sum_{n\geq 0} r_i t^i \,:\, r_0^n = 0 \text{ and } r_1 \text{ invertible}\big\}$$

are representable by $\mathbb{C}[x_i, x_1^{-1}]_{i\geq 0}/(x_o^n)$ and their direct limit is exactly $\mathrm{Aut}\,\mathcal{O}$.

Definition 2.2.17. Let $\mathrm{Der}\,\mathcal{O}$ be the Lie algebra functor of $\mathrm{Aut}\,\mathcal{O}$. Defined as

$$\mathrm{Der}\,\mathcal{O}(R) := \ker(\mathrm{Aut}\,\mathcal{O}(R[\epsilon]) \to \mathrm{Aut}\,\mathcal{O}(R))$$

Therefore an element $x \in \mathrm{Der}\,\mathcal{O}(R)$ is expressed $\rho_\epsilon(t) = t + \epsilon\rho(t)$ for an arbitrary $\rho(t) = \sum_{i\geq 0} r_i t^i$. Indeed the only conditions for $\rho_\epsilon(t)$ to belong in $\mathrm{Aut}\,\mathcal{O}(R[\epsilon])$ are that ϵr_0 is nilpotent which is always true since $\epsilon^2 = 0$ and that $1 + \epsilon r_1$ is invertible, which is true for any $r_1 \in R$ (the inverse is $1 - \epsilon r_1$.

Proposition 2.2.5. $\mathrm{Der}\,\mathcal{O}$ *is representable and isomorphic to* $\mathbb{C}[x_i]$. *The bracket on* $\mathrm{Der}\,\mathcal{O}$ *is expressed as follows:*

$$[t + \epsilon r(t), t + \epsilon s(t)] = t + \epsilon\big(r(t)s'(t) - r'(t)s(t)\big)$$

Proof. The first part of the statement follows from the above discussion while the second is a straightforward computation following the definition of the bracket for a general group functor. □

Derivatives

We are now going to define certain derivatives. We will often be interested in $R[[t]]$ points of groups and we wish to make sense of the derivative along t of such points in terms of the Lie algebras.

Let $(t + \epsilon) : \mathrm{Spec}\, R[[t]][\epsilon] \to \mathrm{Spec}\, R[[t]]$ the R-morphism defined by the morphism of R algebra which sends $t \mapsto t + \epsilon$.Then the composition

$$\text{Spec}\, R[[t]] \xrightarrow{\epsilon=0} \text{Spec}\, R[[t]][\epsilon] \xrightarrow{t\mapsto t+\epsilon} \text{Spec}\, R[[t]]$$

is the identity.

Definition 2.2.18. Let G be an algebraic group. And consider a morphism $g(t) : D_R \to G$ or equivalently an element $g \in G(R[[t]])$. We define its derivative with respect to t as the element

$$dg \cdot g^{-1} := g(t+\epsilon)g^{-1}(t)$$

Since $g(t+\epsilon)_{\epsilon=0} = g(t)$ this morphism defines an element of $\mathfrak{g}(R[[t]])$.

Consider now an element $x(t) \in \mathfrak{g}(R[[t]]) \subset G(R[[t]][\epsilon_0])$. We wish to describe its derivative with respect to t in terms of the ordinary derivative on $\mathfrak{g} \otimes R[[t]]$

Proposition 2.2.6. *Consider now an element* $x(t) \in \mathfrak{g}(R[[t]]) \subset G(R[[t]][\epsilon_0])$. *Then the element* $dx(t) \cdot x^{-1}(t) \in \mathfrak{g}(R[[t]][\epsilon])$ *equals to*

$$\epsilon \partial_t x(t)$$

Proof. It is easily computed for GL_n and since we are considering algebraic groups over $\mathbb{C}$ it is sufficient to prove the general case. □

Chapter 3

Vertex Algebras

In this section we introduce vertex algebras and briefly state the basic results of the theory, a more detailed discussion can be found in [Fre07] and in [Kac98]. Next we are going to define a vertex algebra closely related to $\hat{\mathfrak{g}}_k$ we will call it $V_k(\mathfrak{g})$. We will extensively study this algebra and see for instance how the formalism of vertex algebras allows us to produce a large number of central elements of the completed enveloping algebra.

3.1 Formal calculus

We briefly introduce the language of formal calculus. It involves computations on series living in spaces such as $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$, the space of formal power series a coefficients in $\mathbb{C}$ in the variables z, w.

More generally given a vector space U we consider the vector space $U[[z_i^{\pm 1}]]_i$ where i runs from 1 to a certain natural number n. This is certainly a vector space, in the case in which U is an algebra $U[[z_i^{\pm 1}]]_i$ won't be an algebra with the usual multiplication of series since in the formulas infinite sums appear. If U is an algebra we can multiply series in different variables and in any case is possible multiply series with Laurent polynomials so $U[[z_i^{\pm 1}]]_i$ is a $\mathbb{C}[z_i^{\pm 1}]_i$ module. The partial derivatives ∂_{z_i} are also well defined.

Our standard notation for a series $A(z_1, \dots, z_n) \in U[[z_i^{\pm 1}]]$ will be

$$A(z_1, \dots, z_n) = \sum A_{j_1,\dots,j_n} z_1^{-j_1-1} \dots z_n^{-j_n-1}$$

We define here the linear map $\int(\cdot \, dz_i) := \mathrm{Res}_{z_i=0}(\cdot \, dz_i) : U[[z_j^{\pm 1}]] \to U[[z_j^{\pm 1}, \hat{z_i}^{\pm 1}]]$

$$\int A(z_1, \dots, z_n) dz_i := \sum A_{j_1,\dots,j_{i-1},0,j_{i+1},\dots,j_n} z_1^{-j_1-1} \dots z_{i-1}^{-j_{i-1}-1} z_{i+1}^{-j_{i+1}-1} \dots z_n^{-j_n-1}$$

3.1.1 The formal delta function

We introduce now a very important power series: $\delta(z-w) \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. We define

$$\delta(z-w) := \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$$

here $z-w$ is just a notation. It satisfies the following basic properties:

- For any series $A(z) \in U[[z^{\pm 1}]]$ the product $A(z)\delta(z-w)$ makes sense in $U[[z^{\pm 1}, w^{\pm 1}]]$ and the following equation holds:
$$A(z)\delta(z-w) = A(w)\delta(z-w)$$
- The partial derivatives satisfy $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$
- The formal delta function has some nice behavior with respect to the polynomials $(z-w)^n$. In particular we have that $(z-w)\delta(z-w) = 0$, $(z-w)\partial_w^{n+1}\delta(z-w) = (n+1)\partial_w^n\delta(z-w)$ and therefore $(z-w)^n\partial_w^n\delta(z-w) = n!\delta(z-w)$
- Let $A(z)$ be any formal series in $U[[z^{\pm 1}]]$. Then $A(z)\delta(z-w)$ is a well defined series in $U[[z^{\pm 1}, w^{\pm 1}]]$ and
$$\int A(z)\delta(z-w)dz = A(w)$$

The following proposition holds and will be crucial in the next paragraphs.

Proposition 3.1.1. *The kernel of the multiplication by $(z-w)^N$ in $U[[z^{\pm 1}, w^{\pm 1}]]$ is the $U[[w^{\pm 1}]]$ span of $\delta(z-w), \dots, \partial_w^{N-1}\delta(z-w)$. In addition the coefficients $\gamma_i(w)$ in the expression*

$$x = \sum_{i=0}^{N-1} \gamma_i(w)\partial_w^i\delta(z-w)$$

for an element x in this kernel are unique.

Proof. We prove the firs part of the statement by induction on N. For $N=1$ consider $A(z,w) = \sum_{n,m} A_{n,m} z^{-n-1}w^{-n-1}$ which is killed by $z-w$. Then the following condition on the coefficients holds: $A_{n+1,m} - A_{n,m+1} = 0$. We deduce that the coefficients are constant on the diagonals: $n+m = i+j$ implies that $A_{n,m} = A_{i,j}$. Let $B_k := A_{n,m}$ for any choice of $n+m=k$ and let $B(w) := \sum B_k w^{-k-1}$ then a straightforward computation shows that

$$A(z,w) = B(w)\delta(z-w)$$

Now suppose that we have $A(z,w) \in \ker(\cdot(z-w)^{N+1})$ by the inductive hypothesis we have

$$(z-w)A(z,w) = \sum_{i=0}^{N-1} \gamma_i(w)\partial_w^i\delta(z-w)$$

By the properties cited above we therefore have

$$(z-w)\big(A(z,w) - \sum_{i=0}^{N-1}\frac{1}{i+1}\gamma_i(w)\partial_w^{i+1}\delta(z-w)\big) = 0$$

Finally by the first step of the induction there exists a $\gamma(w) \in U[[w^{\pm 1}]]$ such that

$$A(z,w) = \gamma(w)\delta(z-w) + \sum_{i=1}^{N} \frac{1}{i}\gamma_{i-1}(w)\partial_w^i \delta(z-w)$$

To prove uniqueness notice that given two series $B(w), C(w)$ we have

$$B(w)\delta(z-w) = C(w)\delta(z-w) \implies B(w) = C(w)$$

Therefore if $\sum_{i=0}^{N-1} \gamma_i(w)\partial_w^i \delta(z-w) = \sum_{i=0}^{N-1} \gamma_i'(w)\partial_w^i \delta(z-w)$ multiplying by $(z-w)^{N-1}$ we get $\gamma_{N-1}(w) = \gamma_{N-1}(w)'$ and then we may proceed by induction. □

3.1.2 Fields

We now focus on the case in which $U = \operatorname{End} V$ for another vector space V, we study a particular class of power series which are called fields.

Definition 3.1.1 (Fields)**.** A series $A(z) \in \operatorname{End} V[[z^{\pm 1}]]$ is said to be a **field** if for every $v \in V$ the evaluation of $A(z)$ on v is a Laurent series:

$$A(z)v = \sum_n A_n(v) z^{-n-1} \in V((z)) \qquad \text{equivalently } A_n v = 0 \text{ for } n \gg 0$$

If V is a $\mathbb{Z}$ graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ we have the usual notions of homogeneous elements of V as well as homogeneous elements of $\operatorname{End} V$. $\varphi \in \operatorname{End} V$ is said to homogeneous of degree m if $\varphi(V_n) \subseteq V_{n+m}$. A formal series $A(z) = \sum A_n z^{-n-1}$ is said to be of **conformal dimension** m if A_n is homogeneous of degree $m-n-1$ for every $n \in \mathbb{Z}$. Note that if the gradation on V is bounded from below then any conformal series is automatically a field.

Remark 3.1.1. If $A(z)$ is a field of conformal dimension m then $\partial_z A(z)$ is a field of conformal dimension $m+1$. More generally $\partial_z^n A(z)$ has conformal dimension $m+n$.

Definition 3.1.2 (Normally ordered product)**.** Given two fields $A(z), B(z)$ we define their normally ordered product as

$$:A(z)B(z): := A(z)_+ B(z) + B(z)A(z)_-$$

where $A(z)_+$ stands for the nonnegative part (in the variable z) of $A(z)$ while $A(z)_-$ stands for the negative part.

Even if in the computation of the coefficients appear infinite sums of endomorphism, it is easy to check that they converge in an algebraic sense: for any $v \in V$ the evaluation of the above series involves only finite sums. So the normally ordered product is a well defined element of $\operatorname{End} V[[z^{\pm 1}]]$ and it's not hard to check that it is actually a field.

We define the normally ordered product of multiple series from right to left:

$$:A(z)B(z)C(z): :=: A(z)(:B(z)C(z):):$$

This definition appears to be a little arbitrary, but we will shortly see that it arises quite naturally in the context of vertex algebras.

3.2 Vertex Algebras: definition and basic properties

We start giving a definition about fields.

Definition 3.2.1 (Local Fields)**.** Two fields $A(z), B(z) \in \operatorname{End} V[[z^{\pm 1}]]$ are said to be **mutually local** if there exist a large enough natural number N such that

$$(z-w)^N [A(z), B(w)] = 0$$

Definition 3.2.2 (Vertex Algebra)**.** A **vertex algebra** is a vector space V equipped with the following additional datum:

1. A vector $|0\rangle \in V$ (the vacuum vector)
2. An endomorphism $T : V \to V$ (the translation operator)
3. A linear map $Y(\cdot, z) : V \to \operatorname{End} V[[z^{\pm 1}]]$ with image contained in the subspace of fields

$$A \in V \mapsto Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$$

   The field $Y(A, z)$ will be called the vertex operator associated with A while the endomorpshisms A_n will be called the Fourier coefficients of the vertex operator.

This datum should also satisfy the following axioms:

1. $Y(|0\rangle, z) = \mathrm{id}_V$
2. $Y(A, z)|0\rangle \in A + zV[[z]]$
3. $T|0\rangle = 0$
4. $[T, Y(A, z)] = \partial_z Y(A, z)$ or equivalently $[T, A_n] = -nA_{n-1}$
5. (Locality) For any two vectors $A, B \in V$ the associated vertex operators $Y(A, z), Y(B, z)$ are mutually local fields

The definition looks a little bit cumbersome but we will shortly see how the vertex operators encapsulate a lot of information about the commutation relations of the Fourier coefficients in a very compact way.

Notice that the axioms for a vertex algebra make sense for any module V over a commutative ring A. We will not pursue this point of view too deeply, even if it will come in handy from time to time to consider vertex algebras over commutative rings such as $\mathbb{C}[x]$.

Most of the vertex algebras we will encounter have a natural $\mathbb{Z}$ grading. We therefore give here the definition of what a $\mathbb{Z}$-graded vertex algebra is.

Definition 3.2.3. A vertex algebra V is said to be $\mathbb{Z}$-graded (or $\mathbb{Z}_+$ graded if the gradation is concentrated in positive degree). If the vector space V is $\mathbb{Z}$ graded and so it admits a decomposition $V = \oplus_{n \in \mathbb{Z}} V_n$ and in addition:

1. The vacuum vector $|0\rangle$ has degree 0;

2. The translation operator T has degree 1;

3. For any $A \in V$ homogeneous of degree n (i.e. $A \in V_n$) the field $Y(A,z) = \sum_m A_m z^{-m-1}$ has conformal dimension n, that is to say

$$\deg A_m = n - m - 1$$

   Because of this shifting we will sometimes adopt the following notation: if $A \in V_n$ we will write $Y(A,z) = \sum_n A_m z^{-m-n}$ so that $\deg A_m = -m$.

Now that we know what a vertex algebra is we define some categorical notions.

Definition 3.2.4. We list some definitions concerning morphism, direct sums and tensor products.

- A **morphism** of vertex algebras between two vertex algebras $(V_1, |0\rangle_1, T_1, Y_1)$, $(V_2, |0\rangle_2, T_2, Y_2)$ is a linear map $\varphi : V_1 \to V_2$ that preserves the structures. In particular we require that $\varphi(|0\rangle_1) = |0\rangle_2$, that $\varphi \circ T_1 = T_2 \circ \varphi$, and that $\varphi(A)_n \varphi(B) = \varphi(A_n B)$ for all $n \in \mathbb{Z}$. This last condition may be rephrased as follows:

$$\varphi \circ Y(A,z) = Y(\varphi(A), z) \circ \varphi$$

  This gives us the notion of a category.

- The **kernel** and the **image** of a morphism of vertex algebras are defined as the corresponding vector spaces, they carry natural structures of vertex algebras;

- A vertex subalgebra W of V is a subspace which contains $|0\rangle$, which is stable under the action of T and for which for any $A, B \in W$ the elements $A_n B$ are still in W. Equivalently it is the image of an injective morphism $W \to V$;

- Given two vertex algebras V_1, V_2 their **direct sum** is naturally defined as $V_1 \oplus V_2$ as a vector space, $|0\rangle = |0\rangle_1 + |0\rangle_2$, $T = T_1 + T_2$ and $Y = Y_1 + Y_2$. This may be checked to be a product in the category of vertex algebras;

- Given two vertex algebras V_1, V_2 their **tensor product** is defined as follows. As a vector space it is $V_1 \otimes V_2$, $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$, $T = T_1 \otimes \mathrm{id} + \mathrm{id} \otimes T_2$ and finally $Y(A \otimes B, z) := Y(A,z) \otimes Y(B,z)$. This defines the structure of a vertex algebra and it may be checked that is the coproduct in the category of vertex algebras;

We will give other notions and constructions like the one of ideal and quotient after we have developed a little further the theory.

As an example we will focus for a brief moment on commutative vertex algebras, we will see that the category of commutative vertex algebras is equivalent to the category of commutative algebras equipped with a derivation.

Definition 3.2.5. A vertex algebra V is said to be **commutative** or **abelian** if for any $A, B \in V$ we have $[A_n, B_m] = 0$ for every n, m, or equivalently

$$[Y(A,z), Y(B,w)] = 0$$

Another characterization is the following.

Proposition 3.2.1. *A vertex algebra* V *is commutative if and only if for any* $A \in V$ *we have*

$$Y(A, z) \in \operatorname{End} V[[z]]$$

Proof. If for any $A \in V$ the condition $Y(A, z) \in \operatorname{End} V[[z]]$ is satisfied we obtain that $[Y(A, z), Y(B, w)] \in \operatorname{End} V[[z, w]]$ which by the locality axiom is killed by $(z - w)^N$ for a sufficiently large N. It's not hard to see that in the space $V[[z, w]]$ the multiplication by $(z - w)^N$ is injective, and therefore $[Y(A, z), Y(B, w)]$ must be 0. So V is commutative.

On the other hand assume that V is commutative. Then applying the vacuum vector to the equation $Y(A, z)Y(B, w) = Y(B, w)Y(A, z)$ we obtain

$$Y(A, z)(B + wV[[w]]) = Y(B, w)(A + zV[[z]])$$

Since the series are equal and the one on left hand side has only non negative powers of w while the one on the right hand side has only non negative powers of z they must belong to $V[[z, w]]$. In particular the coefficients in $Y(A, z)B$ polar in z are 0, that is to say that $A_n B = 0$ for any $B \in V$ and for any $n \geq 0$, but this is another way to say that $A_n = 0$ for $n \geq 0$ which implies $Y(A, z) \in \operatorname{End} V[[z]]$. □

Now consider a commutative algebra V (with unity) equipped with a derivation T. One can define a structure of vertex algebra as follows: as a vector space we take V, for the vacuum vector we take the unity 1, while we consider the derivation T as the translation operator. Finally we define the vertex operators as

$$Y(A, z) := \sum_{n \geq 0} \frac{1}{n!} \operatorname{mult}(T^n A) z^n$$

where $\operatorname{mult}(T^n A)$ stands for the endomorphism given by the multiplication by $T^n A$. It is a straightforward to check that this datum satisfy the axioms of a vertex algebra, which of course i commutative.

One the other hand consider a commutative vertex algebra V. Define a bilinear product on V by

$$A \cdot B := A_{-1} B$$

Since V is commutative we have $AB = A_{-1}B = A_{-1}B_{-1}|0\rangle = B_{-1}A_{-1}|0\rangle = B_{-1}A = BA$ so this product is symmetric (and associative) and it is of course bilinear since A_{-1} is linear as well as the association $A \mapsto A_{-1}$. Taking $|0\rangle$ as the unity we obtain a commutative algebra.

One may check that the translation operator is a derivation with respect to this product and that the vertex operators are of the form $Y(A, z) = \sum \frac{1}{n!}(T^n A)_{-1} z^n$ which is the same formula that we used to define the structure of a vertex algebra on a commutative algebra.

The following proposition holds:

Proposition 3.2.2. *The above construction defines an equivalence between the category of commutative vertex algebras and the category of commutative unital algebras equipped with a derivation.*

Given an arbitrary vertex algebra (not necessarily commutative) we can still define what its center is.

Definition 3.2.6. Let V be a vertex algebra. We define its center $\zeta(V)$ as the following subspace

$$\zeta(V) := \left\{ A \in V : [Y(A, z), Y(B, w)] = 0 \text{ for all } B \in V \right\}$$

We will check that it is an abelian vertex subalgebra of V after we have developed some of the theory.

3.2.1 First properties of Vertex Algebras

We are going to give here an outlook of some of the basic properties of vertex algebras. We will follow [Fre07], omitting some of the proofs.

We begin with a simple but very useful fact.

Proposition 3.2.3. *Let V be a vertex algebra and $A \in V$. Then*

$$Y(A,z)|0\rangle = e^{zT}A = \sum_{n\geq 0} \frac{z^n}{n!} T^n A$$

Proof. By the translation axiom we have that $[T, Y(A,z)] = \partial_z Y(A,z)$ in particular, since $T|0\rangle = 0$ we obtain

$$TY(A,z)|0\rangle = [T, Y(A,z)]|0\rangle = \partial_z(Y(A,z)|0\rangle)$$

This is a differential equation for $Y(A,z)|0\rangle$. Since we know that the constant term of $Y(A,z)|0\rangle$ is A, we can find recursively using the above equation all the other coefficients and a straightforward computation shows that they are exactly the ones of $e^{zT}A$. □

Lemma 3.2.1 (Translation)**.** *In any vertex algebra we have*

$$e^{wT}Y(A,z)e^{-wT} = Y(A,z+w)$$

where in the right hand side negative powers of $Z+w$ are expanded assuming that w/z is small, that is to say, in positive powes of w/z.

Proof. It is an easy computation to prove that

$$e^{wT}Y(A,z)e^{-wT} = \sum_{n\geq 0} (\operatorname{ad} T)^n \big(Y(A,z)\big)\frac{w^n}{n!} = \sum_{n\geq 0} \partial_z^n \big(Y(A,z)\big)\frac{w^n}{n!}$$

And this is just the formal Taylor expansion of $Y(A,z+w)$ in positive powers of w. □

This lemma states that the exponentiation of T gives us the translation operator $z \mapsto z+w$.

Proposition 3.2.4 (Skew Symmetry)**.** *In any vertex algebra the following identity holds:*

$$Y(A,z)B = e^{zT}Y(B,-z)A$$

Proof. By locality we know that there exists an N such that

$$(z-w)^N Y(A,z)Y(B,w)|0\rangle = (z-w)^N Y(B,w)Y(A,z)|0\rangle$$

and this is actually an equality in $V[[z,w]]$. We compute this expression as follows using the translation property we just proved.

$$\begin{aligned}
(z-w)^N Y(A,z)e^{wT}B &= (z-w)^N Y(B,w)e^{zT}A \\
(z-w)^N Y(A,z)e^{wT}B &= (z-w)^N e^{zT}Y(B,w-z)A \\
z^N Y(A,z)B &= z^N e^{zT}Y(B,-z)A \\
Y(A,z)B &= e^{zT}Y(B,-z)A
\end{aligned}$$

We justify the above calculations. The left hand side of the second row, as in the translation proposition must be understood as expanded in positive powers of z/w. By comparison with the expression on the right hand side we find that actually only positive powers of w appear and therefore the factor $(z-w)^N$ must cancel out all negative powers of $(z-w)$ in $Y(B, w-z)A$. This allows us to set $w = 0$ and the proof is concluded. $\square$

As an application of skew symmetry we are now able to state a consistent definition of an ideal in a vertex algebra.

Definition 3.2.7 (Ideals)**.** A subspace I of a vertex algebra V is called an *deal* if it is preserved by the action of T and if for every $A \in I$, every $B \in V$ and every $n \in \mathbb{Z}$ the element $A_n B$ still belongs to I. Skew symmetry tells us that ideals are actually two sided (i.e. if $A \in I, B \in V, n \in \mathbb{Z}$ we have also $B_n A \in I$). Indeed by skew symmetry we have $Y(B,z)A = e^{zT} Y(A,-z)B \in I((z))$. This allows us to put a natural structure of vertex algebra on the quotient space V/I.

3.2.2 More on Locality

We review in this section the locality axiom, restating it in another fashion. This point of view allows us to state another fundamental property of vertex algebra which is called **associativity**. This property will lead to the crucial basic feature of vertex algebras: the operator product expansion (OPE).

Given a vector space V the vector space of formal series $V[[z^{\pm 1}, w^{\pm 1}]]$ contains various notable subspaces. The firs one is the subspace of regular series, which do not contain negative powers of z and w: $V[[z,w]]$. The next two spaces we are interested in are $V((z))((w))$ and $V((w))((z))$ of series with negative powers of w (resp. z) bounded from below, their intersection is the space of series with negative powers bounded from below for both z and w: the space $V[[z,w]][z^{-1}, w^{-1}]$.

Consider for a moment the case in which $V = \mathbb{C}$ so the spaces $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ are actually fields, and the ring $\mathbb{C}[[z,w]][z^{-1}, w^{-1}]$ is contained in both of them. In particular since $z - w$ is invertible in both these fields we obtain two embeddings:

$$\begin{array}{ccc} & & \mathbb{C}((z))((w)) \\ & \nearrow & \\ \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}] & & \\ & \searrow & \\ & & \mathbb{C}((w))((z)) \end{array}$$

They are different maps, viewing the two fields contianed in the same vector space $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$. In particular the upper embedding amounts to expanding $(z-w)^{-1}$ in positive powers of w/z while, the lower one corresponds to the expansion of $(z-w)^{-1}$ in positive powers of z/w.

We have analogous embeddings in the case of $V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ which is a localization of $V[[z,w]]$ viewed as a $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ module.

Now lets turn back to the situation of vertex algebra and lets focus on the locality axiom. Pick three vectors A, B, C. By locality there exists a natural number N such that

$$(z-w)^N Y(A,z)Y(B,w)C = (z-w)^N Y(B,w)Y(A,z)C$$

The left hand side belongs to the space $V((z))((w))$ while the right hand side of the equation lies in $V((w))((z))$, being equal they must belong to the intersection of this spaces: $V[[z,w]][z^{-1},w^{-1}]$. This leads to the following reformulation of the locality axiom.

Proposition 3.2.5. *In any vertex algebra V for any triple of vectors $A,B,C \in V$ the expressions $Y(A,z)Y(B,w)C$ and $Y(B,w)Y(A,z)C$ are expression of one same element in $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ in the spaces $V((z))((w))$ and $V((w))((z))$ respectively.*

Proof. Since $(z-w)^N Y(A,z)Y(B,w)C \in V[[z,w]][z^{\pm 1},w^{\pm 1}]$ the element $Y(A,z)Y(B,w)C$ must be in the image of the embedding $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}] \to V((z))((w))$, the analogous statement is true for $Y(B,w)Y(A,z)C$. Using the equality above, and reading it in the space $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ where we can divide by $z-w$ we conclude the proof.' □

3.2.3 Associativity

We saw in the previous section that the locality axiom may be rephrased as an equality of the elements $Y(A,z)Y(B,w)C$ and $Y(B,w)Y(A,z)C$ in the space $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ which embeds in two different ways in $V((z))((w))$ and $V((w))((z))$. There is a third natural space in which we can embed $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ that is $V((w))((z-w))$.

The following proposition holds.

Theorem 3.2.1 (Associativity)**.** *Consider a vertex algebra V and three vectors $A,B,C \in V$. Then the elements*

$$Y(A,z)Y(B,w)C \quad Y(B,w)Y(A,z)C \quad Y(Y(A,z-w)B,w)C$$

are expansions of the same unique element in $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$ in the corresponding spaces $V((z))((w)), V((w))((z)), V((w))((z-w))$.

Here $Y(Y(A,z-w)B,w)C$ stands for the series

$$\sum_n Y(A_n B,w)C\frac{1}{(z-w)^{n+1}}$$

Proof. See [Fre07], theorem 2.3.3. □

It is useful to think the associativity property as an equality

$$Y(A,z)Y(B,w) = \sum_n Y(A_n B,w)(z-w)^{-n-1} \tag{3.1}$$

This is very convenient since relates a product of two vertex operators with a linear combination of other veretx operators but we have to be careful. Appliying both sides to a vector C we do obtain the convergence of the series but they still are not exactly equal: they are expansion of the same element of $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$. This equality, with the above understanding in the following way, is called the **operator product expansion** or **OPE** for short.

We are now ready to see some of the most important corollaries of associativity which are some of the most essential features of vertex algebras. We start with a technical lemma.

Lemma 3.2.2. *Let $\varphi(z), \psi(w)$ be two fields on the same space. Then the following are equivalent:*

1.

$$[\varphi(z), \psi(w)] = \sum_{i=0}^{N-1} \frac{1}{i!} \gamma_i(w) \partial_w^i \delta(z-w)$$

2.

$$\varphi(z)\psi(w) = \sum_{i=1}^{N-1} \gamma_i(w) \left(\frac{1}{(z-w)^{i+1}} \right)_{|z|>|w|} + :\varphi(z)\psi(w): \text{ and}$$
$$\psi(w)\varphi(z) = \sum_{i=1}^{N-1} \gamma_i(w) \left(\frac{1}{(z-w)^{i+1}} \right)_{|w|>|z|} + :\varphi(z)\psi(w):$$

Where by $(\frac{1}{(z-w)^{i+1}})_{|z|>|w|}$ we mean its expansion in positive powers of w/z.

Proof. This is a simple computation. A detailed proof may be found in [Fre07], lemma 2.3.4. □

Now our vertex operators satisfy the first condition of the lemma, being mutually local, and moreover thanks to the OPE formula we are able to compute the polar coefficients in $(z-w)$ of $Y(A,z)Y(B,w)$ we obtain the following theorem.

Theorem 3.2.2. *Given two vectors A, B in a vertex algebra V, the commutation relation of their vertex operators may be written as follows:*

$$[Y(A,z), Y(B,w)] = \sum_{n \geq 0} \frac{1}{n!} Y(A_n B, w) \partial_w^n \delta(z-w) \tag{3.2}$$

While for any couple of vectors A, B and any number $n \in \mathbb{Z}$ we have

$$Y(A_n B, z) = \frac{1}{(-n-1)!} :\partial_z^{-n-1} Y(A,z) \cdot Y(B,z): \quad \text{if } n < 0 \tag{3.3}$$
$$Y(A_n B, z) = \int (w-z)^n [Y(A,w), Y(B,z)] dw \quad \text{if } n \geq 0 \tag{3.4}$$

Proof. It can be found in [Fre07] section 2.3.4 'corollaries of associativity'. □

Keeping in mind the first part of the theorem we will rewrite the OPE formula as follows:

$$Y(A,z)Y(B,w) \sim \sum_{n \geq 0} \frac{Y(A_n B, w)}{(z-w)^{n+1}}$$

This will be a shorter notation to keep in mind that the commutation relations between the two fields $Y(A,z)$ and $Y(B,w)$ is encoded in the polar part (i.e. with negative powers of $z-w$) on the right hand side.

We are now able to see how normally ordered product was not an 'ad hoc' definition but it arises quite naturally in the contex of vertex algebras. We obtain also the following useful corollaries.

Corollary 3.2.1. *The following hold*

1. *For any $A \in V$ we have*
$$Y(TA, z) = \partial_z Y(A, z)$$
2. *For any m-tuple of vectors $A^1, \dots, A^m$ and negative numbers $n_1, \dots, n_m$ we have*
$$Y(A^1_{n_1} \dots A^m_{n_m}|0\rangle, z) = \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} : \partial_z^{-n_1-1} Y(A^1, z) \cdot \dots \cdot \partial_z^{-n_m-1} Y(A^m, z)$$
3. *The span of the Fourier coefficients $A_n \in \mathrm{End}\, V$ is a Lie subalgebra of $\mathrm{End}\, V$, the bracket is given by the followig formula:*
$$[A_n, B_m] = \sum_{k \geq 0} \binom{n}{k} (A_k B)_{n+m-k}$$
Where we use the enlarged definition of binomial which allows n to be negative:
$$\binom{n}{k} := \frac{n(n-1)\dots(n-k+1)}{k!} \text{ if } k > 0; \qquad \binom{n}{0} = 1$$

Proof. Part 1 follows from the second statement of theorem 3.2.2 applied considering $B = |0\rangle$ and $n = -2$ and finally noticing that $A_{-2}|0\rangle = TA$, this is easy since $TA = TA_{-1}|0\rangle = [T, A_{-1}]|0\rangle = A_{-2}|0\rangle$. Part 2 is an simply an iterative application of the second part of theorem 3.2.2. While the third statement is follows from expanding the terms in formula 3.2. □

We are also now ready to prove that the center of a vertex algebra is actually a subalgebra.

Corollary 3.2.2. *The following equality holds:*
$$\zeta(V) = \{A \in V : A_n B = 0 \text{ for all } n \geq 0 \text{ and } B \in V\}$$
In particular Given a vertex algebra V, its center $\zeta(V)$ is a subalgebra of V (which is of course abelian).

Proof. Consider formula 3.2. We have already proved in proposition 3.1.1 that the formal series $\partial_w^i \delta(z - w)$ are $V[[w^{\pm 1}]]$ linearly independent. From this we deduce that $[Y(A, z), Y(B, w)] = 0$ if and only if $Y(A_n B, z) = 0$ for all $n \geq 0$. Finally we remark that the map Y is injective, since $(Y(A, z)|0\rangle)_{z=0} = A$. So $Y(A_n B, z) = 0$ if and only if $A_n B = 0$, this proves the first part of the corollary.

Now to see that $\zeta(V)$ is a vertex subalgebra of V we start by remarking that since $Y(|0\rangle, z) = \mathrm{id}_V$ we certainly have $|0\rangle \in \zeta(V)$. Moreover since $Y(TA, z) = \partial_z Y(A, z)$ it easily follows that $\zeta(V)$ is invariant under the action of T.

Notice that the condition $A_n B = 0$ for all $B \in V$ and for all $n \geq 0$ is equivalent to ask that $Y(A, z) \in \mathrm{End}\, V[[z]]$. Finally we have to check that if A and B are central so it is $A_n B$ for any $n \in \mathbb{Z}$. This is certainly true for $n \geq 0$ since $A_n B = 0$ by hypothesis. For $n < 0$ we use the second part theorem 3.2.2.
$$Y(A_n B, z) = \frac{1}{(-n-1)!} : \partial_z^{-n-1} Y(A, z) \cdot Y(B, z) :$$
Since both series are regular (without polar parts in z) we have
$$: \partial_z^{-n-1} Y(A, z) \cdot Y(B, z) := \left(\partial_z^{-n-1} Y(A, z)\right) Y(B, z) \in \mathrm{End}\, V[[z]]$$
and therefore $Y(A_n B, z) \in \mathrm{End}\, V[[z]]$ so $A_n B$ is central as well. □

It is also possible to define the centralizer of an element S.

Definition 3.2.8. Let $S \in V$. Define the **centralizer** of S in V as the subspace

$$Z(S) := \{A \in V : [Y(A,z), Y(S,w)] = 0\} = \{A \in V : A_n S = 0 \text{ for all } n \geq 0\} = \{A \in V : Y(A,z)S \in V[[z]]\}$$

It is a vertex subalgebra of V.

Proof. The equalities above, as in the case of $\zeta(V)$, easily follow from theorem 3.2.2. Since it is quite clear that $|0\rangle \in Z(S)$ and that $Z(S)$ is invariant under the action of T, to prove that $Z(S)$ is a vertex subalgebra of V it suffices to consider two elements $A, B \in Z(S)$ and show that for any $k \in \mathbb{Z}$ the product $A_k B \in Z(S)$. Using theorem 3.2.2 we see that for $k < 0$

$$\begin{aligned} Y(A_k B, z)S &= \frac{1}{(-k-1)!} : \partial_z^{-k-1} Y(A,z) \cdot Y(B,w) : S \\ &= \frac{1}{(-k-1)!} \left(\partial_z^{-k-1} Y(A,z)_+ Y(B,z)S + Y(B,z) \partial_z^{-k-1} Y(A,z)_- S \right) \end{aligned}$$

which is clearly in $V[[z]]$. While for $k \geq 0$

$$Y(A_k B, z)S = \int \Big((w-z)^k \big(Y(A,w)Y(B,z)S + Y(B,z)Y(A,w)S \big) \Big) dw$$

which again is easily seen to be in $V[[z]]$. □

The characterization we gave in theorem 3.2.2 of the vertex operators naturally leads to the following theorem which is a good starting point to construct vertex algebras.

Theorem 3.2.3. *(Strong Reconstruction) Let* V *be a vector space,* $|0\rangle \in V$ *a vector and* T *an endomorphism of* V*. Let*

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a_n^\alpha z^{-n-1}$$

where α *runs over an ordered set* I*, be a collection of fields on* V *such that:*

1. $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$*;*
2. $T|0\rangle = 0$ *and* $a^\alpha(z)|0\rangle \in V[[z]]$*, we will call* $a^\alpha := a_{-1}^\alpha |0\rangle$*;*
3. *For any* $\alpha, \beta \in I$ *the fields* $a^\alpha(z)$ *and* $a^\beta(z)$ *are mutually local;*
4. *The lexicographically ordered monomials* $a_{-n_1-1}^{\alpha_1} \dots a_{-n_m-1}^{\alpha_n} |0\rangle$ *with* $n_i \geq 0$ *span* V*.*

Then the formula

$$Y(a_{-n_1-1}^{\alpha_1} \dots a_{-n_m-1}^{\alpha_n} |0\rangle, z) = \prod_{i=1}^{i=m} \frac{1}{n_i!} : \partial_z^{n_1} a^{\alpha_1}(z) \dots \partial_z^{n_m} a^{\alpha_m}(z) :$$

defines a vertex algebra structure on V *such that* $|0\rangle$ *is the vacuum vector,* T *the translation operator and* $Y(a^\alpha, z) = a^\alpha(z)$ *for all* $\alpha \in I$*. Moreover, this is the unique vertex algebra structure on* V *satisfying conditions 1,2,3,4 and such that* $Y(a^\alpha, z) = a^\alpha(z)$*.*

We are going to give some example of vertex algebras though we will limit ourselves to the vertex algebras that are the most interesting for us: the Virasoro algebra and the algebra $V_k(\hat{\mathfrak{g}})$ associated to the affine algebra $\hat{\mathfrak{g}}_k$.

3.3 The Virasoro Algebra

Consider $K = \mathbb{C}((t))$ and let

$$\mathrm{Der}\, K = \mathbb{C}((t))\partial_t$$

be the Lie algebra of continuous derivation of K, with the Lie bracket given by the usual formula

$$[f(t)\partial_t, g(t)\partial_t] = \big(f(t)g'(t) - f'(t)g(t)\big)\partial_t$$

We consider a central extension of $\mathrm{Der}\, K$ which we will denote by Vir the Virasoro algebra.

Definition 3.3.1. The bilinear map

$$\bigwedge^2 \mathrm{Der}\, K \to \mathbb{C} \qquad (f(t)\partial_t, g(t)\partial_t) \mapsto -\frac{1}{12}\mathrm{Res}_{t=0}\big(f(t)g'''(t)dt\big)$$

is a cocycle in $H^2(\mathrm{Der}\, K, \mathbb{C})$ and therefore defines a central one dimensional extension Vir of $\mathrm{Der}\, K$ which satisfies the exact sequence

$$0 \to \mathbb{C}C \to \mathrm{Vir} \to \mathrm{Der}\, K \to 0$$

As a vector space $\mathrm{Vir} = \mathrm{Der}\, K \oplus \mathbb{C}C$, C is central and the bracket is given by

$$[f(t)\partial_t, g(t)\partial_t]_{\mathrm{Vir}} = \big(f(t)g'(t) - f'(t)g(t)\big)\partial_t - \frac{C}{12}\mathrm{Res}_{t=0}\big(f(t)g'''(t)dt\big)$$

If we consider the topological generators $L_n := -t^{n+1}\partial_t$ $n \in \mathbb{Z}$ we obtain the following relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}C \tag{3.5}$$

In the definition the factor $\frac{1}{12}$ is of course inessential, but it is present for notational and historical reasons. In what follows we will define a vertex algebra closely related to Vir, we will see that this construction in a certain sense generalizes to the construction of $V_k(\hat{\mathfrak{g}})$.

3.3.1 The vertex Virasoro algebra

Consider the Lie subalgebra $\mathrm{Der}\,\mathcal{O} \oplus \mathbb{C}C \subset \mathrm{Vir}$ where $\mathrm{Der}\,\mathcal{O}$ is the Lie subalgebra consisting of derivations $f(t)\partial_t$ with $f(t) \in \mathbb{C}[[t]]$. Consider also its one dimensional module $\mathbb{C}_c$ where $\mathrm{Der}\,\mathcal{O}$ acts trivially while C acts by the multiplication of $c \in \mathbb{C}$.

Define a Vir module inducing the representation just presented:

$$\mathrm{Vir}_c := \mathrm{Ind}_{\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}C}^{\mathrm{Vir}}\mathbb{C}_c = U(\mathrm{Vir}) \otimes_{U(\mathrm{Der}\,\mathcal{O}\oplus\mathbb{C}C)} \mathbb{C}_c$$

This is of course a Vir module and by the Poincaré-Birkhoff-Witt theorem (which we will refer to as the PBW theorem from now on) it has a basis given by the monomials

$$L_{n_1} \dots L_{n_m}|0\rangle \quad \text{with } n_1 \leq \dots \leq n_m < -1$$

Where $|0\rangle$ is a fixed generator of $\mathbb{C}_c$. We define a structure of $\mathbb{Z}_+$ graded vertex algebra on Vir_c, we will extensively use the reconstruction theorem presented in the previous section (3.2.3).

- ($\mathbb{Z}_+$ grading) We set $\deg L_{n_1} \dots L_{n_m}|0\rangle = -\sum_i n_i$. Notice that since in Vir we have the relations $[L_0, L_n] = -nL_n$ and since $L_0|0\rangle = 0$ the operator L_0 acts exactly as the grading operator;

- (Translation operator) Set the translation to be $T := L_{-1}$. Notice that the action of T is completely determined by the properties that in Vir the relations $[L_{-1}, L_n] = (-n-1)L_{n-1}$ hold and that $L_{-1}|0\rangle$;

- (Vertex Operators) We set

$$Y(L_{-2}|0\rangle, z) = T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

By the reconstruction theorem and since the monomials $L_{n_1} \dots L_{n_m}|0\rangle$ span Vir_c to obtain a vertex algebra we only have to check that $T(z)$ is local with itself and that $[T, Y(L_{-2}|0\rangle, z)] = \partial_z Y(L_{-2}|0\rangle, z)$. Both of these are straightforward calculations and we omit them. It turns out that

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3 \delta(z-w) + 2T(w)\partial_w \delta(z-w) + \partial_w T(w) \cdot \delta(z-w)$$

and therefore $(z-w)^4[T(z), T(w)] = 0$. The number c is known as the **central charge**.

The Virasoro algebra comes in the picture numerous times. A very important class of vertex algebras is equipped with an action of Vir where C acts by scalar multiplication by a certain central charge c. We are particularly interested in such algebras where the action comes by internal symmetries, which is to say that there is a vector $\omega \in V$ whose Fourier coefficients generates the action of the Virasoro algebra. This leads to the following definition.

Definition 3.3.2. A $\mathbb{Z}$ graded vertex algebra V is said to be **conformal** of central charge c if there is a non-zero vector $\omega \in V_2$ such that the Fourier coefficients L_n^V of the corresponding vertex operator

$$Y(\omega, z) = \sum_n L_n^V z^{-n-2}$$

satisfy the commutation relation of the Virasoro algebra with central charge c and if in addition we have $\deg = L_0^V$ and $T = L_{-1}^V$. Note that with respect with our standard definition $L_n^V = \omega_{n+1}$

A vertex algebra may be equipped with various structures of conformal vertex algebra. A first example is the Virasoro vertex algebra itself, taking the conformal vector to be $\omega = L_{-2}|0\rangle$. In this case the conformal vector is unique since $(\mathrm{Vir}_c)_2$ is one dimensional and L_{-2} is the only scalar multiple of L_{-2} which satisfies all the desired properties (for instance $\lambda(L_{-2}|0\rangle)_1 = \lambda \deg$). We state a very useful lemma.

Lemma 3.3.1. *A $\mathbb{Z}_+$ graded vertex algebra V is conformal of central charge c if and only if it contains a nonzero vector $\omega \in V_2$ such that the Fourier coefficients of the corresponding vertex operator*

$$Y(\omega, z) = \sum_n L_n^V z^{-n-2}$$

satisfy the following conditions: $L_{-1}^V = T, L_0^V = \deg$ and $L_2^V \omega = \frac{c}{2}|0\rangle$. Moreover in the case this conditions are satisfied there exists a unique morphism of vertex algebras $\mathrm{Vir}_c \to V$ such that $L_{-2}|0\rangle \mapsto \omega$.

Proof. We need to show the following OPE

$$Y(\omega,z)Y(\omega,w) = \frac{Y(T\omega,w)}{(z-w)} + 2\frac{Y(\omega,w)}{(z-w)^2} + \frac{c}{2}\frac{1}{(z-w)^4}$$

which amounts to show that

$$L^V_{-1}\omega = T\omega \quad L^V_0\omega = 2\omega \quad L^V_1\omega = 0 \quad L^V_2\omega = \frac{c}{2}|0\rangle \quad L^V_n\omega = 0 \text{ for } n > 2$$

The last equation is true because V is $\mathbb{Z}_+$ graded and $L^V_n\omega$ has negative degree for $n > 2$, all the other equations are true by hypothesis, except the third one.

To prove it denote by $\gamma(w) := Y(L^V_1\omega, w)$ by the OPE formula we find

$$[Y(\omega,z), Y(\omega,w)] = \frac{c}{12}\partial^3_w\delta(z-w) + \gamma(w)\partial^2_w\delta(z-w) + 2Y(\omega,w)\partial_w\delta(z-w) + \partial_w Y(\omega,w)\cdot\delta(z-w)$$

Now we consider the same expression with z replaced by w and vice versa. Using the fact that $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$ and summing the two equations we find

$$0 = (\gamma(w) + \gamma(z))\partial^2_w\delta(z-w)$$

□

3.4 The Verma module $V_k(\hat{\mathfrak{g}})$

We are now going to define a vertex algebra $V_k(\hat{\mathfrak{g}})$ closely related to $\hat{\mathfrak{g}}_k$, the vacuum Verma module. The subspace $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ is a Lie subalgebra of $\hat{\mathfrak{g}}_k$. Analogously to what we have done for the Virasoro algebra we consider the one dimensional module $\mathbb{C}|0\rangle$ for $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{g}[[t]]$ acts trivially while $\mathbf{1}$ acts as the identity.

Next we consider the induced module

$$V_k(\hat{\mathfrak{g}}) := \mathrm{Ind}\,^{\hat{\mathfrak{g}}_k}_{\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1}}\mathbb{C}|0\rangle$$

One can easily check that this is a smooth module for $\hat{\mathfrak{g}}_k$. Let J^a be an ordered basis for $\mathfrak{g}$ (this will be our standard notation from now on) and denote by $J^a_n := J^a \otimes t^n \in \hat{\mathfrak{g}}_k$, these are topological generators for $\hat{\mathfrak{g}}_k$ subject to the relations

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\kappa(J^a, J^b)\delta_{n,-m}\mathbf{1}$$

By the Poincarè-Birkhoff-Witt theorem $V_k(\hat{\mathfrak{g}})$ has a basis consisting of the lexicographically ordered monomials

$$J^{a_1}_{n_1}\dots J^{a_m}_{n_m}|0\rangle \quad \text{with } n_1 \le n_2 \le \dots \le n_m < 0 \text{ and if } n_i = n_{i+1} \text{ then } a_i \le a_{i+1}$$

We are going to define the structure of a $\mathbb{Z}_+$ graded vertex algebra on $V_k(\hat{\mathfrak{g}})$.

- We take $|0\rangle$ as the vacuum vector;
- We set the grading to be $\deg J^{a_1}_{n_1}\dots J^{a_m}_{n_m}|0\rangle := -\sum n_i$

- As the translation operator we consider the operator $-\partial_t$ acting on $\hat{\mathfrak{g}}_k$, more concretely we define the operator T through the conditions

$$[T, J^a_n] = nJ^a_{n-1} \qquad T|0\rangle = 0$$

- For the vertex operators we define

$$Y(J^a_{-1}, z) = J^a(z) := \sum_n J^a_n z^{-n-1}$$

By the reconstruction theorem, as in the case of the Virasoro algebra, we only have to check that the vertex operators $J^a(z)$ are mutually local with each other and that $[T, Y(J^a_{-1}|0\rangle, z) = \partial_z Y(J^a_{-1}, z)$. This are both quite easy calculations, in particular it turns out that

$$[J^a(z), J^b(w)] = [J^a, J^b](w)\delta(z-w) + \kappa(J^a, J^b)\partial_w \delta(z-w) \tag{3.6}$$

As the $\mathbb{Z}_+$ grading is concerned it is easy to see that $Y(J^a_{-1}|0\rangle, z) = \sum_n J^a_n z^{-n-1}$ has conformal dimension equal to 1 as desired. As remarked before if $A(z)$ is a field of conformal dimension m its derivative $\partial^n_z A(z)$ is a field of conformal dimension $m+n$, so the fields $Y(J^a_n|0\rangle, z) = \frac{1}{(-n-1)!}\partial^n_z Y(J^a_{-1}|0\rangle, z)$ have conformal dimension n as desired. An easy induction shows that $Y(J^{a_1}_{n_1} \dots J^{a_m}_{n_m}|0\rangle, z)$ has conformal dimension $-\sum n_i$. Finally the operator T is clearly of degree 1 while $|0\rangle$ is of degree 0. All axioms of a $\mathbb{Z}_+$ graded vertex algebra are hence satisfied.

We are now going to study more in detail the vertex algebra $V_k(\hat{\mathfrak{g}})$ as k varies, we are going to define the Segal-Sugawara operators (or just Sugawara operators for short). Studying these operators we will see the first appearance of the critical value $k = -1/2$, indeed the Sugawara operators will be central only for $k = -1/2$.

3.4.1 The Segal-Sugawara Operators

Consider the killing form $\kappa_{\mathfrak{g}}$ on $\mathfrak{g}$. Since it is non degenerate given a basis J^a of $\mathfrak{g}$ we may consider the dual basis J_a with respect to the killing form. It is defined by the relations $\kappa_{\mathfrak{g}}(J^a, J_b) = \delta_{a,b}$. Consider the element

$$S := \sum_a J^a_{-1} J_{a,-1}|0\rangle \in V_k(\hat{\mathfrak{g}})$$

Definition 3.4.1. (Segal-Sugawara Operators) Given a simple Lie algebra $\mathfrak{g}$ consider the vacuum Verma module $V_k(\hat{\mathfrak{g}})$ and the element $S \in V_k(\hat{\mathfrak{g}})$ just defined. The operators S_n

$$Y(S, z) = \sum_n S_n z^{-n-2} =: S(z)$$

are called the **Segal-Sugawara Operators**.

The definition of the element S gets inspiration from the Casimir element of the standard universal enveloping algebra $U(\mathfrak{g})$. Similarly to the Casimir element S does not depend on the choice of the basis J^a (but if we change the associative form with which we calculate the dual basis S will be modified by a scalar). The Casimir element being central, it is reasonable to think that the Sugawara operators commute at least with the action of $\hat{\mathfrak{g}}_k$ on $V_k(\hat{\mathfrak{g}})$. This though, is not always true as the following proposition shows.

Proposition 3.4.1. *We have the following OPE*

$$J^a(z)S(w) \sim (k+\frac{1}{2})\frac{J^a(w)}{(z-w)^2}$$

In particular the following commutation relations hold

$$[J^a_m, S_n] = (k+\frac{1}{2})mJ^a_{n+m} \tag{3.7}$$

*And therefore the Sugawara operators commute with the action of $\hat{\mathfrak{g}}_k$ if and only if $\kappa = -\frac{1}{2}\kappa_{\mathfrak{g}}$ whis will be called **the critical value**.*

Proof. This is just a simple computation. By the OPE formula we have

$$J^a(z)S(w) \sim \sum_{n\geq 0}\frac{Y(J^a_n S, w)}{(z-w)^{n+1}}$$

Since $V_k(\hat{\mathfrak{g}})$ is $\mathbb{Z}_+$ graded, S is of degree 2 and J^a_n is of degree $-n$ we only need to compute $J^a_n S$ for $n = 0, 1, 2$. We recall here the commutation relations in $\hat{\mathfrak{g}}_k$:

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\kappa(J^a, J^b)\delta_{n,-m}$$

- ($n = 0$) Since $[J^a_0, J^b_{-1}] = [J^a, J^b]_{-1}$ and since $J^a_0|0\rangle = 0$ we have

  $$J^a_0 S = \frac{1}{2}\sum_b([J^a, J^b]_{-1}J_{b,-1} + J^b_{-1}[J^a, J_b]_{-1})|0\rangle$$

  This corresponds to take the commutator with the Casimir element. The same proof of the centrality of the Casimir element applied in our situation provides us with $J^a_0 S = 0$;

- ($n = 1$) As before $J^a_1|0\rangle = 0$ and $[J^a_1, J^b_{-1}] = [J^a, J^b]_0 + \kappa(J^a, J^b)\mathbf{1}$. We have:

  $$J^a_1 S = \frac{1}{2}\sum_b\Big(\big([J^a, J^b]_0 + \kappa(J^a, J^b)\big)J_{b,-1} + J^b_{-1}\big([J^a, J_b]_0 + \kappa(J^a, J_b)\mathbf{1}\big)\Big)|0\rangle$$

  The expression

  $$\frac{1}{2}\sum_b\big(\kappa(J^a, J^b)J_{b,-1} + \kappa(J^a, J_b)J^b_{-1}\big)|0\rangle$$

  is exactly equal to $kJ^a_{-1}|0\rangle$. Indeed since J_b is the dual basis of J^b with respect with the killing form $\kappa_{\mathfrak{g}}$ we have

  $$\frac{1}{2}\sum_b\kappa(J^a, J^b)J_b = \frac{k}{2}\sum_b\kappa_{\mathfrak{g}}(J^a, J^b)J_b = \frac{k}{2}J^a$$

  The same is true for $\frac{1}{2}\sum_b\kappa(J^a, J_b)J^b$.

  Next we focus on the remaining term

  $$\frac{1}{2}\sum_b\Big([J^a, J^b]_0 J_{b,-1} + J^b_{-1}[J^a, J_b]_0\Big)|0\rangle = \frac{1}{2}\sum_b[J^a, J^b]_0 J_{b,-1}|0\rangle = \frac{1}{2}\sum_b[[J^a, J^b]J_b]_{-1}|0\rangle$$

We must therefore calculate the sum $\sum_b [[J^a, J^b], J_b]$. We claim it is equal to J^a. Notice first that this is exactly the action of the Casimir element $\sum_b J_b J^b$ through the adjoint representation. Since $\mathfrak{g}$ is simple and the Casimir element is central it must act like a scalar. To determine this scalar we compute the trace

$$\mathrm{tr}\big(\sum_b J_b J^b\big) = \sum_b \mathrm{tr}(\mathrm{ad}\,(J_b)\mathrm{ad}\,(J_b)) = \sum_b \kappa_{\mathfrak{g}}(J_b, J^b) = \dim \mathfrak{g}$$

So $\sum_b J_b J^b$ acts as the identity on the adjoint representation and $\sum_b [[J^a, J^b]J_b] = J^a$ as desired. To sum up we proved that $J_1^a S = (k + \frac{1}{2}) J_{-1}^a |0\rangle$;

- ($n = 2$) As always we have $J_2^a|0\rangle = 0$ and $[J_2^a . J_{-1}^b] = [J^a, J^b]_1$ we have to compute

$$\frac{1}{2}\sum_b \left([J^a, J^b]_1 J_{b,-1} + J_{-1}^b [J^a, J_b]_1\right)|0\rangle = \frac{1}{2}\sum_b [J^a, J^b]_1 J_{b,-1}|0\rangle = \frac{1}{2}\sum_b \left([[J^a, J^b]J_b]_0 + \kappa([J^a, J^b], J_b)\mathbf{1}\right)|0\rangle$$

The first term of this last sum acts like 0 on $|0\rangle$ so all that we are left to compute is

$$\sum_b \kappa([J^a, J^b], J_b)$$

Since S does not depend on the choice of the basis, we may as well assume that J^b is an orthonormal basis so that $J_b = J^b$. Using the associativity of κ we find out that $\kappa([J^a, J^b], J^b) = \kappa(J^a, [J^b, J^b] = 0$ and the sum is 0.

The last part of the proposition, the statement that $[J_m^a, S_n] = (k + \frac{1}{2}) m J_{n+m}^a$ follows from a simple expansion of the OPE we just calculated. □

We just found out that the Sugawara Operators commute with the action of $\hat{\mathfrak{g}}_k$ for $k = k_c = -\frac{1}{2}$. We remark here that some of these operators are actually 0. Indeed for $n \geq -1$ since $Y(S, z)|0\rangle \in V[[z]]$ we have that $S_n|0\rangle = 0$, in addition S_n commutes with the action of $\hat{\mathfrak{g}}_k$ so it must be identically 0.

We investigate now some of the properties of the Sugawara Operators away from the critical level. So, in what follows, consider $k \neq \frac{1}{2}$. Normalize S with this assumption as follows:

$$\tilde{S} := \frac{1}{k + 1/2} S$$

From formula 3.7 with this new normalization the following commutation relations hold:

$$[\tilde{S}_n, J_m^a] = -m J_{n+m}^a$$

This is a very nice formula. In particular it shows that $S_0 = \deg$ and that $S_{-1} = T$.

Recalling that $J_n^a = J^a \otimes t^n$ we see that the action of $\tilde{S}_n$ looks like the action of $-t^{n+1}\partial_t$. We want to see if it is true that the commutations relations of the operators $\tilde{S}_n$ are the same as the operators $-t^{n+1}\partial_t$. We will see that this is almost true: the $\tilde{S}_n$ do not generate an action of $\mathrm{Der}\, K$ but they do generate an action of its central extension Vir.

Proposition 3.4.2. *The following OPE relation hold:*

$$\tilde{S}(z)\tilde{S}(w) \sim \frac{\partial_w \tilde{S}(w)}{(z-w))} + \frac{2\tilde{S}(w)}{(z-w)^2} + \frac{\frac{k}{k+1/2}\dim\mathfrak{g}/2}{(z-w)^4}$$

We denote the constant term of the factor $(z-w)^{-4}$ by $c_k/12$. In particular the $\tilde{S}_n$ generate an action of the Virasoro algebra with central charge c_k. $V_k(\hat{\mathfrak{g}})$ for $k \neq -1/2$ is therefore a conformal vertex algebra of central charge c_k.

Proof. It is sufficient to calculate the OPE

$$\tilde{S}(s)\tilde{S}(w) \sim \sum_{n\geq -1} \frac{Y(S_n S, w)}{(z-w)^{n+2}}$$

the shifting of indices is due to our shifted definition of the $\tilde{S}_n$. Since S_n has degree n and S has degree 2 we can restrict ourselves to $n = -1, 0, 1, 2$, recall that for $n \geq -1$ we have $\tilde{S}_n|0\rangle = 0$. We proceed with the calculations applying the commutation relations

$$[\tilde{S}_n, J^a_m] = -mJ^a_{n+m}$$

- ($n = -1$) We must calculate $\tilde{S}_{-1}\frac{1}{2}\frac{1}{k+1/2}\sum_b J^b_{-1}J_{b,-1}|0\rangle$ this is equal to

  $$\frac{1}{2}\frac{1}{k+1/2}\sum_b \Big([\tilde{S}_{-1}, J^b_{-1}]J_{b,-1} + J^b_{-1}[\tilde{S}_{-1}, J_{b,-1}]\Big)|0\rangle = \frac{1}{2}\frac{1}{k+1/2}\sum_b \big(J^b_{-2}J_{b,-1} + J^b_{-1}J_{b,-2}\big)|0\rangle$$

  which is just $T(\tilde{S})$

- ($n = 0$) As in the previous point we calculate

  $$\frac{1}{2}\frac{1}{k+1/2}\sum_b \Big([\tilde{S}_0, J^b_{-1}]J_{b,-1}+J^b_{-1}[\tilde{S}_0, J_{b,-1}]\Big)|0\rangle = \frac{1}{2}\frac{1}{k+1/2}\sum_b \big(J^b_{-1}J_{b,-1}+J^b_{-1}J_{b,-1}\big)|0\rangle = 2\tilde{S}$$

- ($n = 1$) We omit the first passage which is always the same, we obtain

  $$\frac{1}{2}\frac{1}{k+1/2}\sum_b \big(J^b_0 J_{b,-1} + J^b_{-1}J_{b,0}\big)|0\rangle = \frac{1}{2}\frac{1}{k+1/2}\sum_b [J^b, J_b]_{-1}|0\rangle$$

  which is 0 as we can choose $J^b = J_b$;

- ($n = 2$)

  $$\frac{1}{2}\frac{1}{k+1/2}\sum_b \big(J^b_1 J_{b,-1}+J^b_{-1}J_{b,1}\big)|0\rangle = \frac{1}{2}\frac{1}{k+1/2}\sum_b \big([J^b, J_b]_0+\kappa(J^b, J_b)\mathbf{1}\big)|0\rangle = \frac{k}{k+1/2}\dim\mathfrak{g}/2|0\rangle$$

□

We obtained that for any $k \neq -1/2$ the vertex algebra $V_k(\hat{\mathfrak{g}})$ is a conformal algebra. Informally, the algebra $V_{k_c}(\hat{\mathfrak{g}})$ may be viewed as the limit for $k \to k_c$ of the vertex algebras $V_k(\hat{\mathfrak{g}})$ for $k \neq -1/2$, we may wander if the conformal structures 'pass to the limit' and generate at least an action of the Virasoro algebra.

The following definition arises quite naturally.

Definition 3.4.2. A a vertex algebra V is said to be **quasi-conformal** if it is equipped with a $\mathrm{Der}\,\mathcal{O}$ action which satisfies the following conditions:

- For any $A \in V$ and any $n \geq -1, m \in Z$

$$[L_n, A_m] = \sum_{k \geq -1} \binom{n+1}{k+1} (L_k A)_{n+m-k}$$

- The operator $L_{-1} = -\partial_t$ acts like the operator T;
- L_0 acts semisimply with integer eigenvalues;
- The Lie subalgebra $\mathrm{Der}_+\mathcal{O} = t\mathbb{C}[[t]]\partial_t$ acts locally nilpotently.

These axioms, in particular the first one, emulate the behaviour of a conformal algebra. The main difference is that the action of $\mathrm{Der}\,\mathcal{O}$ does not need to be generated by a conformal vector, neither we ask it to be part of a $\mathrm{Der}\,K$ action.

Remark 3.4.1. A quasi conformal vertex algebra for which $L_0|0\rangle = |0\rangle$ is automatically $\mathbb{Z}$ graded. Indeed consider L_0 as the grading operator, this is well defined since L_0 acts semisiply with integer eigenvalues. The translation operator has degree 1 since $[L_0, L_{-1}] = L_{-1}$. On the other hand, considering an homogeneous vector $A \in V_m$ (i.e $L_0 A = mA$) we have that

$$[L_0, A_n] = \sum_{k \geq -1} \binom{1}{k+1} (L_k A)_{n-k} = (L_{-1}A)_{n+1} + (L_0 A)_n = (TA)_{n+1} + mA_n = (m-n-1)A_n$$

so A_n has degree $m-n-1$ as desired.

Notice that $\hat{\mathfrak{g}}_k$ has a natural $\mathrm{Der}\,\mathcal{O}$ action: namely the one induced from the natural action on $\mathfrak{g}((t))$. Since this action preserves the subalgebra $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ we have that it induces an action on the algebra $V_k(\hat{\mathfrak{g}})$. For k away from the critical level, this action is described by the Sugawara operators. At the critical level we still have

Proposition 3.4.3. *The natural action of* $\mathrm{Der}\,\mathcal{O}$ *on the vertex algebra* $V_{k_c}(\mathfrak{g})$ *makes it a quasi conformal vertex algebra.*

Proof. Consider the lie algebra $L\mathfrak{g}[\mathbf{k}] = L\mathfrak{g} \otimes \mathbb{C}[\mathbf{k}]$ where $\mathbb{C}[\mathbf{k}]$ is the free polynomial algebra in the variable $\mathbf{k}$. Consider the $\mathbb{C}[\mathbf{k}]$ bilinear cocycle on $L\mathfrak{g}[\mathbf{k}]$ defined by

$$c\big(X \otimes f(t) \otimes p(\mathbf{k}), Y \otimes g(t) \otimes q(\mathbf{k})\big) := p(\mathbf{k})q(\mathbf{k})\kappa_{\mathfrak{g}}(X,Y) \int f(t)g'(t)dt$$

denote by $\hat{\mathfrak{g}} = L\mathfrak{g}[\mathbf{k}] \oplus \mathbb{C}[\mathbf{k}]\mathbf{1}$ the central extension obtained with this cocycle. It is a $\mathbb{C}[\mathbf{k}]$ Lie algebra.

Consider the module

$$V(\hat{\mathfrak{g}}) := \operatorname{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[[t]][\mathbf{k}]\oplus\mathbb{C}[\mathbf{k}]\mathbf{1}} \mathbb{C}[\mathbf{k}]|0\rangle$$

where $\mathfrak{g}[[t]][\mathbf{k}]$ acts trivially while $\mathbf{1}$ acts as the identity (all actions are $\mathbb{C}[\mathbf{k}]$-linear). By the PBW theorem $V(\hat{\mathfrak{g}})$ has a $\mathbb{C}[\mathbf{k}]$ basis given by the lexicographically ordered monomials

$$J^{a_1}_{n_1} \dots J^{a_m}_{n_m}|0\rangle$$

as in the case of $V_k(\hat{\mathfrak{g}})$. We define $\mathbb{C}[\mathbf{k}]$-linear vertex operators in the same way we did with $V_k(\hat{\mathfrak{g}})$. We obtain a $\mathbb{C}[\mathbf{k}]$ vertex algebra whose reduction to $\mathbf{k} - k = 0$ is isomorphic to $V_k(\hat{\mathfrak{g}})$.

Let S be defined as in the case of $V_k(\hat{\mathfrak{g}})$

$$S := \frac{1}{2} \sum_a J^a_{-1} J_{a,-1}|0\rangle \in V(\hat{\mathfrak{g}})$$

then the vertex operators S_n for $n \in \mathbb{Z}$ defined by $Y(S, z) = \sum_{n\in\mathbb{Z}} S_n z^{-n-2}$ satisfy the following relations:

$$[S_n, J^a_m] = -(\mathbf{k} + 1/2) m J^a_{n+m}$$

As well as the ordinary vertex algebra relations

$$[S_n, A_m] = \sum_{k\geq -1} \binom{n+1}{k+1} (S_k A)_{n+m-k}$$

(the shift of the indices is due to the fact that the S_n are shifted with respect to the usual notation).

Now the action of $\operatorname{Der}\mathcal{O}$ on $L\mathfrak{g}$ induces a $\mathbb{C}[\mathbf{k}]$ linear action on $L\mathfrak{g}[\mathbf{k}]$ and hence a $\mathbb{C}[\mathbf{k}]$ linear action on $V(\hat{\mathfrak{g}})$ whose specialization to $\mathbf{k} = k$ coincide with the $\operatorname{Der}\mathcal{O}$ action on $V_k(\hat{\mathfrak{g}})$. The commutation relations

$$[L_n, J^a_m] = -m J^a_{n+m}$$

uniquely determine the operators L_n and therefore we find that

$$S_n = (\mathbf{k} + 1/2) L_n$$

These implies that the expressions

$$[L_n, A_m] \quad \text{and} \quad \sum_{k\geq -1} \binom{n}{k} (L_k A)_{n+m-k}$$

are equal after multiplying by $(\mathbf{k} + 1/2)$. These expressions live in $\operatorname{End}_{\mathbb{C}[\mathbf{k}]} V(\hat{\mathfrak{g}})$ which is a free $\mathbb{C}[\mathbf{k}]$ module and hence torsion free. The expressions above are therefore equal (it is not necessary anymore to multiply by $(\mathbf{k}+1/2)$) and specializing to $\mathbf{k} = -1/2 = k_c$ we get the desired statement. □

This concludes the properties that we wanted to explore in this chapter. In the following chapter we will investigate the relation between $V_k(\hat{\mathfrak{g}})$ and the completed enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Chapter 4

The relationship between $V_\kappa(\mathfrak{g})$ and $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$

In this chapter we investigate the relationship between the vertex algebra $V_k(\hat{\mathfrak{g}})$ and the completed enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. We will see that there is a way to associate to any vertex operator in $V_k(\hat{\mathfrak{g}})$ a sequence of elements in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. This association preserves the commutation relations in particular central elements in $V_k(\hat{\mathfrak{g}})$ will produce a lot of central elements $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$. Everything will be explained more precisely in what follows.

4.1 The Lie algebra associated to a Vertex Algebra

In the previous chapter we presented the commutation formulas

$$[A_n, B_m] = \sum_{k\geq 0} \binom{n}{k} (A_k B)_{n+m-k} \qquad [T, A_n] = -nA_{n-1}$$

which are valid in $\mathrm{End}\, V$ and prove that the span of the Fourier coefficients in $\mathrm{End}\, V$ for a Lie algebra with the usual commutator. We are now going to generalize this result: we will define $U(V)$ the Lie algebra of 'formal Fourier coefficients'. It is topologically spanned by elements of the form $A_{[n]}$ with $A \in V$ which are the analogues of the Fourier coefficient A_n. This will be much larger than the actual algebra of Fourier coefficient for instance we may have $A_n = 0$ but $A_{[n]} \neq 0$. This property is going to be very important.

Definition 4.1.1. Let $U(V)$ be the vector space defined as

$$U(V) := \frac{V \otimes_{\mathbb{C}} \mathbb{C}((t))}{\mathrm{Im}\, \partial} \qquad \text{where } \partial = T \otimes \mathbf{1} + \mathbf{1} \otimes \partial_t$$

and equip U(V) with the topology induced by the subspaces $U(V)_n = \mathrm{Im}\,(V \otimes t^n\mathbb{C}[[t]] \to U(V))$. This makes $U(V)$ into the completion of $U'(V) := V \otimes \mathbb{C}[t, t^{-1}]/\mathrm{Im}\, \partial$ under the topology generated by the subspaces $U'(V)_n := \mathrm{Im}(V \otimes t^n\mathbb{C}[t] \to U'(V))$. For $A \in V$ and $n \in \mathbb{Z}$ denote by

$$A_{[n]} := [A \otimes t^n]$$

This is a set of topological generators for $U(V)$ and a set of generators for $U'(V)$. Finally define a bilinear map

$$[.,.] : U'(V)^{\otimes 2} \to U'(V) \qquad [A_{[n]}, B_{[m]}] := \sum_{k \geq 0} \binom{n}{k} (A_k B)_{[n+m-k]}$$

where $A_k B$ is the usual k-product on V.

The bilinear map defined above may be checked to be continuous with respect to the topology defined above. We will denote by $[.,.]$ its extension to $U(V)$. We want to check that $[.,.]$ is actually a Lie bracket on $U(V)$. By continuity it is enough to check it on $U'(V)$ and therefore we will do our calculations on elements of the form $A_{[n]}$.

Proposition 4.1.1. *The formula above is well defined and it induces a Lie bracket on* $U(V)$. *The natural map*

$$Y : U(V) \to \operatorname{End} V \qquad [A \otimes f(t)] \mapsto \int Y(A, z) f(z) dz$$

is an homomorphism of Lie algebras.

Proof. Let's check first that $[.,.]$ is well defined on $U'(V)$. The formula above certainly defines a bilinear map $\left(V \otimes \mathbb{C}[t, t^{-1}]\right)\right)^{\otimes 2} \to V \otimes \mathbb{C}[t, t^{-1}]$. We need to check that $[\partial(v), w] \in \operatorname{Im} \partial$ and that $[v, \partial(w)] \in \operatorname{Im} \partial$ for any $v, w \in V \otimes \mathbb{C}[t, t^{-1}]$.

- ($[v, \partial(w)] \in \operatorname{Im} \partial$) As stated before we can restrict ourselves to the elements of the form $v = A_{[n]}$ and $w = B_{[m]}$ we have

$$[A_{[n]}, \partial(B_{[m]})] = [A_{[n]}, (TB)_{[m]} + mB_{[m-1]}] = \sum_{k \geq 0} \binom{n}{k} \big(A_k(TB))_{[n+m-k]} + m(A_k B)_{[n+m-k-1]}\big)$$

  This is easily checked to be equal to

$$\partial \Bigg(\sum_{k \geq 0} \binom{n}{k} (A_k B)_{[n+m-k]} \Bigg)$$

- ($[\partial(v), w] \in \operatorname{Im} \partial$) The expression

$$[\partial A_{[n]}, B_{[m]}] = \sum_{k \geq 0} \binom{n}{k} \big((TA)_k B)_{[n+m-k]}\big) + \binom{n-1}{k} n (A_k B)_{[n+m-k-1}$$

  may be checked to be 0.

Let's prove now that this bilinear map gives us actually a Lie bracket. We are going to consider first the subspace $U'(V)_0 = V / \operatorname{Im} T = \operatorname{Im}(V \to U'(V))$ given by $A \mapsto A_{[0]}$. Notice first that this subspace is preserved by the bilinear map $[.,.]$. Indeed $[A_{[0]}, B_{[0]}] = (A_0 B)_{[0]}$.

From the skew symmetry property

$$Y(A, z)B = e^{zT} Y(B, -z)A$$

it is quite clear that the following equality holds in V

$$A_0 B = -B_0 A + T(\dots)$$

This easily implies that [., .] is alternating:

$$[A_{[0]}, B_{[0]}] = (A_0 B)_{[0]} = -(B_0 A)_{[0]} + (T(\dots))_{[]} = -(B_0 A)_{[0]} = -[B_{[0]}, A_{[0]}]$$

To prove the Jacobi identity consider the equality

$$[A_0, B_0] = (A_0 B)_0$$

which is easily derived from the OPE formula. We next compute

$$[A_{[0]}, [B_{[0]}, C_{[0]}]] = (A_0(B_0 C))_{[0]} = \big([A_0, B_0]C + B_0 A_0 C\big)_{[0]} = \big((A_0 B)_0 C + B_0(A_0 C)\big)_{[0]}$$

which is exactly $[[A_{[0]}, B_{[0]}], C_{[0]}] + [B_{[0]}, [A_{[0]}, C_{[0]}]]$. We proved that the subspace $V/\operatorname{Im} T$ with this bilinear form is actually a Lie algebra. To deduce that the entire space $U'(V)$ is a Lie algebra with the product we defined consider the vertex algebra $V \otimes \mathbb{C}[t, t^{-1}]$ where we consider $\mathbb{C}[t, t^{-1}]$ as a commutative vertex algebra with the derivation ∂_t. Recall that the vertex operators are given by

$$Y(A \otimes B, z) = Y(A, z) \otimes Y(B, z)$$

In our particular case consider $A \otimes t^n$ and recall that

$$Y(t^n, z) = \sum_{k \geq 0} \frac{z^k}{k!} \operatorname{mult}(\partial_t^k t^n) = \sum_{k \geq 0} \binom{n}{k} \operatorname{mult}(t^{n-k}) z^k$$

So we have

$$(A \otimes t^n)_0 = \sum_{k \geq 0} A_k \otimes \left(\operatorname{mult}(\binom{n}{k} t^{n-k}) \right)$$

In particular if we can identify $U'(V) = U'(V \otimes \mathbb{C}[t, t^{-1}])_0$ since $T_{V \otimes \mathbb{C}[t, t^{-1}]} = \partial$. We find from the above formulas that the bilinear map defined in both cases is actually the same. Since we proved that on $U'(V \otimes \mathbb{C}[t, t^{-1}])_0$ it defines a Lie bracket the same must be true for $U'(V)$. □

Remark finally that the association $V \mapsto U(V)$ is actually functorial. Indeed given a morphism of vertex algebras $\varphi : V \to V'$ is quite easy to see that the morphism

$$U(\varphi) : U(V) \to U(V') \qquad A_{[n]} \mapsto (\varphi(A))_{[n]}$$

Is well defined and an homomorphism of Lie algebras.

We conclude with a fundamental remark regarding the center of the vertex algebras we are considering.

Remark 4.1.1. If $S \in V$ is a central element then the elements $S_{[n]} \in U(V)$ are all central (i.e $[x, S_{[n]}] = 0$ for any $x \in U(V)$).

Proof. Recall that we already proved that an element $S \in V$ is central if and only if for any $A \in V$ we have $A_n S = 0$ for all $n \geq 0$. This implies that for any $A_{[m]} \in V$

$$[A_{[m]}, S_{[n]}] = \sum_{k \geq 0} \binom{n}{k} (A_k S)_{[m+n-k]} = 0$$

Since the $A_{[m]}$ topologically span $U(V)$ and the bracket is continuous the proof is completed. □

4.2 The relationship between $V_\kappa(\hat{\mathfrak{g}})$ and $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$

We are going to define a morphism of Lie algebras $U(V_k(\hat{\mathfrak{g}})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ which sends $(J^a_{-1}|0\rangle)_{[n]} \mapsto J^a_n$.

This remarkable morphism allow us to produce a large number of central elements in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ since any central element $S \in \zeta(V_k(\hat{\mathfrak{g}}))$ will generate central elements in $S_{[n]} \in U(V)$, which in particular commute with the elements $(J^a_{-1}|0\rangle)_{[n]}$. The images of the $S_{[n]}$ commute therefore with the J^a_m, since these (along with **1** which is already central) topologically generate the algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ we obtain that the images of the $S_{[n]}$ are also central in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Start by considering the formal series

$$Y[J^a_{-1}|0\rangle, z] = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

as series with coefficients in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

With the next definition we extend the definition of **fields** to the case of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ in order to exhibit some of the typical vertex algebras calculations in this setting.

Definition 4.2.1. A formal series $a(z) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]$ is called a $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$**-field** if its action by left multiplication induced on the quotients $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ is the one of a field.

The following is an easy verification.

Remark 4.2.1. The formal series $Y[J^a_{-1}|0\rangle, z]$ are $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields. If $a(z) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ is a $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-field then its derivative $\partial_z a(z)$ is a field.

Lemma 4.2.1. *If $a(z), b(z) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ are two $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields then the formal series*

$$\partial^n_w \delta(z-w)_+ a(w)b(z)$$

where $\delta(z-w)_+$ is the positive part of $\delta(z-w)$ with respect to the variable w, is a well defined series in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}, w^{\pm 1}]]$. Its residue with respect to w is therefore a well defined series in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ which in addition on $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-field.

In particular the normally ordered products (calculated in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$)

$$: \partial^n_z a(z) \cdot b(z) :$$

are a well defined formal series in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$ and in addition they are $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields.

Proof. We consider the coefficients of $a(z)$ and $b(z)$ as the endomorphisms of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ induced by left multiplication. The analogous statements for fields on a vector space is easily seen to be true. Therefore the formal series

$$\partial^n_w \delta(z-w)_+ a(w)b(z)$$

Defines a well defined endomorphism of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ for every positive integer $n \geq 0$.

Since the various fields defined on the quotients $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ are compatible with the projections $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_m \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ each coefficient of $\partial^n_w \delta(z-w)_+ a(z)b(w)$ defines an endomorphism of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

Analogously for any $n \geq 0$ and for any $k \in \mathbb{Z}$ the infinite sums appearing as the coefficients of $\partial_w^n \delta(z-w)_+ a(z)b(w)1$ are (by the field condition) actually finite sums in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$, since $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is complete we have that each coefficient of $\partial_w^n \delta(z-w)_+ a(z)b(w)$ defines an element of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

The endomorphism of the coefficients $\partial_w^n \delta(z-w)_+ a(w)b(z)$ on $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_n$ coincide with the left multiplication by these elements and therefore the residue $\int \partial_w^n \delta(z-w)_+ a(w)b(z)dw$ is again a $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-field.

The statement concerning the normally ordered product easily follows from noticing that

$$\int \left(\partial_w^n \delta(z-w)_+ a(w)b(z) - \partial_w^n \delta(z-w)_- b(z)a(w)\right) dw =: \partial_z^n a(z) \cdot b(z) :$$

□

Corollary 4.2.1. *For any $m \geq 1$, any m-tuple of integers $n_1, \ldots, n_m < 0$ and any m-tuple of indices $a_1, \ldots, a_m$ the normally ordered product*

$$: \partial_z^{-n_1-1} Y[J_{-1}^{a_1}|0\rangle, z] \ldots \partial_z^{-n_m-1} Y[J_{-1}^{a_m}|0\rangle, z] :$$

computed in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is again a well defined formal series with coefficients in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

This allows us to extend linearly the map $J_{-1}^a|0\rangle \mapsto Y[J_{-a}^a|0\rangle], z]$ linearly to the whole $V_k(\hat{\mathfrak{g}})$

$$Y[., z] : V_k(\hat{\mathfrak{g}}) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]]$$

$$Y[J_{n_1}^{a_1} \ldots J_{n_m}^{a_m}|0\rangle, z] = \frac{1}{-n_1-1} \cdots \frac{1}{-n_m-1} : \partial_z^{-n_1-1} Y[J_{-1}^{a_1}|0\rangle, z] \ldots \partial_z^{-n_m-1} Y[J_{-1}^{a_m}|0\rangle, z] :$$

This is definition is completely analogous to the one of the vertex operators. There is a big difference though: the coefficients of these new vertex operators are elements of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$!

Theorem 4.2.1. *The linear map $\Phi : U'(V_k(\hat{\mathfrak{g}})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ defined on the generators*

$$A_{[n]} \mapsto \left(Y[A, z]\right)_n$$

where $\left(Y[A, z]\right)_k$ is the k-th coefficient of the series (i.e. the coefficient of z^{-k-1} following our usual notation). Is a well defined continuous linear map and it is an homomorphism of Lie algebras. In particular since $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is complete it induces an homomorphism of Lie algebras

$$\Phi : U(V_k(\hat{\mathfrak{g}})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

Consider for a moment the following diagram:

$$\begin{array}{ccccc} U(V_k(\hat{\mathfrak{g}})) & \xrightarrow{\Phi} & \widetilde{U}_\kappa(\hat{\mathfrak{g}}) & \longrightarrow & \operatorname{End} V_k(\hat{\mathfrak{g}}) \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ U(V_k(\hat{\mathfrak{g}})) & & \longrightarrow & & \operatorname{End} V_k(\hat{\mathfrak{g}}) \end{array}$$

All the maps reported above are homomorphisms of Lie algebras and by how we defined the map Φ this diagram is also commutative. If the morphism $U(V_k(\hat{\mathfrak{g}})) \to \operatorname{End} V_k(\hat{\mathfrak{g}})$ was injective we would already be done, but unfortunately this is false in general. In particular it is false the case we are most interested: $k = k_c$. Indeed we will see that the Sugawara operators $\Phi(S_{[n]})$ are non-zero in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$, but we remarked that for $n \geq -1$ and $k = k_c$ act like 0 on $V_{k_c}(\mathfrak{g})$, so the morphism $U(V) \to \operatorname{End} V_{k_c}(\hat{\mathfrak{g}})$ is not injective.

Consequently the theorem does not follow from the above diagram and we have to work a little bit harder. Its proof will occupy the rest of the section, we will start with some definitions and some technical lemmas.

Definition 4.2.2 (m-product for series)**.** Given two $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ fields (or two fields) $a(z), b(z)$ we define

$$a(w)_m b(w) := \int (z-w)^m [a(z), b(w)] dz \qquad m \geq 0$$

$$a(w)_m b(w) := \frac{1}{(-m-1)!} : \partial_z^{-m-1} a(w) \cdot b(w) : \qquad m < 0$$

In addition notice that for $m \geq 0$ the following equality holds:

$$a(w)_{-m-1} b(w) := \frac{1}{m!} \int \left(\partial_w^m \delta(z-w)_+ a(z) b(w) - \partial_w^m \delta(z-w)_- b(w) a(z) \right) dz$$

Lemma 4.2.2 ($\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ Dong Lemma)**.** *If $a(z), b(z), c(z)$ are mutually local $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields then the fields $a(z)_m b(z)$ and $c(z)$ are mutually local for every $m \in \mathbb{Z}$ as well.*

Proof. This is exactly the same proof as the Dong Lemma regarding the usual notion of fields. A proof can be found in [Fre07], lemma 2.2.3. □

If $a(z)$ and $b(z)$ are also local with respect to each other we know that

$$[a(z), b(w)] = \sum_{m \geq 0} \frac{1}{m!} C_m(w) \partial_w^m \delta(z-w)$$

It is quite clear, using the properties of δ, that

$$C_m(w) = a(w)_m b(w) \qquad m \geq 0$$

We immediately notice that in order to prove that the map $U(V_k(\hat{\mathfrak{g}}) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is an homomorphism of Lie algebra it is enough to prove the following proposition.

Proposition 4.2.1. *For any two vectors $A, B \in V_k(\hat{\mathfrak{g}})$ and for any $m \in \mathbb{Z}$ we have that*

$$Y[A, w]_m Y[B, w] = Y[A_m B, w]$$

Indeed this is clearly implies the theorem since by the above remarks we obtain $[Y[A, z], Y[B, w]] = \sum_{m \geq 0} \frac{1}{m!} Y[A_m B, w] \partial_w^m \delta(z-w)$.

The following lemma can be found also in [Kac98, Theorem 4.1], we restate it in terms of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields.

Lemma 4.2.3 (Kac). *Let V be a collection of mutually local $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$-fields (or fields) which is a vector space and which is closed by all the m-th products then V alongside the datum $|0\rangle = \mathrm{id}_{\widetilde{U}_\kappa(\hat{\mathfrak{g}})}$, $T = \partial_t$ and the vertex operators*

$$Y(a(t), z)b(t) := \sum_{k\in\mathbb{Z}} a(t)_k b(t) z^{-k-1}$$

is a vertex algebra.

Proof. It is not difficult to show that if $(z-w)^N[a(z), b(w)] = 0$ then

$$(z-w)^N[Y(a(t), z), Y(b(t), w)] = 0$$

and this essentially proves locality. The other axioms of a vertex algebra are obvious. □

Using the Dong lemma repeatedly we see that the closure for m-products of the subspace $Y[V_k(\hat{\mathfrak{g}}), z] \subset \widetilde{U}_\kappa(\hat{\mathfrak{g}})[[z^{\pm 1}]$ consists of mutually local $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$fields. By the previous lemma it is a vertex algebra with the above defined structure.

We obtain the following corollary.

Corollary 4.2.2. *Given $A, B, C \in V_k(\hat{\mathfrak{g}})$ and any integers m, n the following equality holds:*

$$Y[A,z]_m(Y[B,z]_n Y[C,z]) = Y[B,z]_n(Y[A,z]_m Y[C,z]) + \sum_{j\geq 0}\binom{m}{j}(Y[A,z]_j Y[B,z])_{m+n-j} Y[C,z]$$

and the skew symmetry formula

$$a(z)_n b(z) = (-1)^{n+1}\left(\sum_{m\geq 0}\frac{1}{m!}\partial_z^m\big(b(z)_{n+m}a(z)\big)\right)$$

Lemma 4.2.4. *Let $n < 0$ denote by $J^a_n(z) := \frac{1}{(-n-1)!}\partial_z^{-n-1}J^a(z)$. Then for any $n, m \geq 0$*

$$[J^a_{-n-1}(z), J^b_{-m-1}(w)] = (-1)^n\frac{1}{n!m!}\left(\sum_{k=0}^m\binom{m}{k}\partial_w^{m-k}[J^a, J^b](w)\partial_w^{k+n}\delta(z-w) + \kappa(J^a, J^b)\partial_w^{n+m+1}\delta(z-w)\right)$$

In particular we have the following expressions for the products $J^a_n(z)_k J^b_m(z)$ for $k \geq 0$.

- *It is 0 for $k < n$ or $k > n+m+1$*
- *It is equal to*

$$(-1)^n\frac{k!}{n!m!}\binom{m}{k-n}\partial_w^{n+m-k}[J^a, J^b](w) = (-1)^n\binom{k}{n}[J^a, J^b]_{k-n-m-1}(w)$$

  for $k \in [n, n+m]$
- *For $k = n+m+1$ it is equal to*

$$(-1)^n\frac{(n+m+1!)}{n!m!}\kappa(J^a, J^b)$$

Proof. The first formula easily follows from taking the derivatives of the equality

$$[J^a(z), J^b(w)] = [J^a, J^b](w)\delta(z-w) + \kappa(J^a, J^b)\partial_w\delta(z-w)$$

While the formulas for the positive products $J^a_n(z)_k J^b_m(z)$ follows from the fact that for any two mutually local fields $a(z), b(z)$

$$[a(z), b(w)] = \sum_{k\geq 0} \frac{1}{k!}\big(a(w)_k b(w)\big)\partial^k_w\delta(z-w)$$

□

The following lemma is the first step to prove proposition 4.2.1. Notice that given a monomial $A \in V_k(\hat{\mathfrak{g}})$ in the J^a_n the following formula

$$Y[J^b_m A, z] = \; : Y[J^b_m|0\rangle, z]Y[A, z] :$$

holds by definition only if the monomial $J^b_m A$ is lexicographically ordered.

Lemma 4.2.5. *For any $n < 0$ for any index a and every $A \in V_k(\hat{\mathfrak{g}})$ the following equality holds:*

$$Y[J^a_n A, z] = \; : Y[J^a_n|0\rangle, z]Y[A, z] : \; = \; : J^a_n(z)Y[A, z] :$$

Proof. By linearity it suffices to prove the statement for A a monomial in the J^b_m. We will do this by induction on d the PBW degree of A.

($d = 1$) We have $A = J^b_m|0\rangle$. If $J^a_n \leq J^b_m$ we are done by definition of Y. So suppose $J^a_n > J^b_m$. We have

$$Y[J^a_n J^b_m|0\rangle, z] = Y[[J^a, J^b]_{n+m}|0\rangle, z] + Y[J^b_m J^a_n|0\rangle, z] = Y[J^a, J^b_{n+m}|0\rangle, z] + : J^b_m(z)J^a_n(z) :$$

Thus we have to prove

$$: J^a_n(z)J^b_m(z) : - : J^b_m(z)J^a_n(z) := [J^a, J^b]_{n+m}(z)$$

and this is a straightforward calculation. And notice that together with Lemma 4.2.4 proves the equality

$$Y(J^a_n|0\rangle, z)_k Y(J^b_m|0\rangle, z) = Y((J^a_n|0\rangle)_k(J^b_{-1}|0\rangle), z) \qquad \forall k \in \mathbb{Z}$$

($d \implies d+1$) Write $A = J^b_m B$ for another monomial B so that $A = J^b_m B$ is lexicographically ordered and $\deg_{PBW} B = d$. Suppose in addition that $J^a_n > J^b_m$ otherwise we are done by definition of Y.

Denote by $A(z)$ and $B(z)$ the vertex operators $Y[A, z]$ and $Y[B, z]$ for simplicity. We need to compute

$$J^a_n(z)_{-1}\big(J^b_m(z)_{-1}B(z)\big) = J^b_m(z)_{-1}\big(J^a_n(z)_{-1}B(z)\big) + \sum_{k\geq 0}\binom{-1}{k}\big(J^a_n(z)_k J^b_m(z)\big)_{k-2}B(z) \tag{4.1}$$

this equality holds by corollary 4.2.2. On the other hand we have

$$Y(J^a_n J^b_m B, z) = Y[J^b_m J^a_n B, z] + Y[[J^a, J^b]_{n+m}B, z]$$

since $n \leq -1$ by assumption, the monomial $[J^a, J^b]_{n+m}B$ is already ordered therefore the second term equals to

$$:[J^a, J^b]_{n+m}(z)B(z):$$

this may be checked to be equal to the second term of formula 4.1 using lemma 4.2.4.

Indeed notice that $J^a_n(z)_k J^b_m(z)$ is of the form $J^c_l(z)$ so we have

$$\big(J^a_n(z)_k J^b_m(z)\big)_{k-2}B(z) = Y(((J^a_n|0\rangle)_k(J^b_m|0\rangle))_{k-2}B, z)$$

and therefore the sum above equals to

$$Y\big([(J^a_n|0\rangle)_{-1}, (J^b_m|0\rangle)_{-1}]B, z\big) = Y([J^a_n, J^b_m]B, z) = Y([J^a J^b]_{n+m}B, z)$$

We are left to compare the terms

$$J^b_m(z)_{-1}\big(J^a_m(z)_{-1}B(z)\big) \quad \text{and} \quad Y[J^b_m J^a_n B, z]$$

in order to prove that they are actually equal we move J^a_n through the factors of B in order to order $J^a_n B$. We obtain $J^a_n B = C + D$ where C is an the lexicographically ordered monomial obtained from $J^a_n B$ and D is the difference $J^a_n B - C$ which is easily seen to have PBW degree $\leq d$.

$$Y[J^b_m J^a_n B, z] = Y[J^b_m C, z] + Y[J^b_m D, z] = \quad :J^b_m(z)C(z): + :J^b_m(z)D(z): \quad =: J^b_m(z)Y[J^a_n B, z]:$$

The second equality follows from the fact that $J^b_m C$ is an ordered monomial and by the inductive hypothesis on $J^b_m D$. Finally, always by the inductive hypothesis we have

$$Y[J^a_n B, z] =: J^a_n(z)B(z):$$

we conclude that

$$J^b_m(z)_{-1}\big(J^a_m(z)_{-1}B(z)\big) = Y[J^b_m J^a_n B, z]$$

and this concludes the proof. □

As a corollary we may immediately see that the function $Y[\cdot, z]$ is well behaved with respect to the operator T.

Lemma 4.2.6. *For any $A \in V_\kappa(\mathfrak{g})$ the following equality holds*

$$Y[TA, z] = \partial_z Y[A, z]$$

Proof. We prove the statement by induction on the PBW degree of A (i.e. its degree as a monomial in the J^a_n). The thesis is of true by definition for $A = J^a_n|0\rangle$. Suppose now that it is true for any monomial of degree $\leq N$ and consider a monomial $J^b_m A$ with $\deg A \leq N$. We have

$$\begin{aligned} Y[TJ^b_m A, z] &= Y[-mJ^b_{m-1}A, z] + Y[J^b_m TA, z] = -m : J^b_{m-1}(z)Y[A, z]: + :J^b_m(z)\partial_z Y[A, z]: \\ &= \partial_z :J^b(z)Y[A, z]:= \partial_z Y[J^b_m A, z] \end{aligned}$$

the second equality follows from the previous lemma while the other are obvious. □

We are now ready to prove proposition 4.2.1.

Proof of proposition 4.2.1. We prove the assertion by induction on the PBW degree of A and B. The case $\deg A = \deg B = 1$ is a straightforward computation and follows from the first formula of lemma 4.2.4.

Now assume that we proved the proposition for $\deg A \leq N$ and $\deg B \leq M$, we are going to prove that the proposition for $\deg B \leq M + 1$. We consider therefore ordered monomials A and $J^b_m B$ with $\deg A \leq N$ and $\deg B \leq M$.

We will write for convenience $Y[A, z] = A(z)$ and $Y[B, z] = B(z)$.

We need to compute

$$A(z)_n\big(J^b_m(z)_{-1}B(z)\big) = J^b_m(z)_{-1}\big(A(z)_n B(z)\big) + \sum_{k\geq 0}\big(A(z)_k J^b_m(z)\big)_{n-1-k}B(z)$$

By the inductive hypothesis $A(z)_k J^b_m(z) = Y(A_k J^b_m|0\rangle)$, in addition it can easily be checked that for $k \geq 0$ $A_k J^b_m$ is a sum of monomials of degree $\leq \deg A$, therefore, always thanks to the inductive hypothesis we have

$$Y(A_k J^b_m|0\rangle, z)_{n-1-k}Y(B, z) = Y\big((A_k J^b_m|0\rangle)_{n-1-k}B, z\big)$$

Summing over $k \geq 0$ and using the analogue identity in $V_k(\hat{\mathfrak{g}})$ We obtain

$$A(z)_n\big(J^b_m(z)_{-1}B(z)\big) = J^b_m(z)_{-1}\big(A(z)_n B(z)\big) + Y([A_n, J^b_m]B, z)$$

Now thanks to lemma 4.2.5 we have

$$A(z)_n\big(J^b_m(z)_{-1}B(z)\big) = Y(J^b_m A_n B, z) + Y([A_n, J^b_m]B, z) = Y(A_n J^b_m B, z)$$

as desired. To conclude the induction it is enough to show that if the statement is true for the couple (A, B) it is also true for $(B.A)$. To see this we consider the skew symmetry formula combined with lemma 4.2.6

$$\begin{aligned} B(z)_n A(z) = \sum_{m\geq 0}\frac{1}{m!}\partial^m_z(A(z)_{m+n}B(z)) &= \sum_{m\geq 0}\frac{1}{m!}\partial^m_z(A_{m+n}B)(z) \\ &= \Bigg(\sum_{m\geq 0}\frac{1}{m!}T^m(A_{m+n}B)\Bigg)(z) = (B_n A)(z) \end{aligned}$$

□

4.2.1 A complete topological algebra associated to a vertex algebra

Now that we proved that the map $U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is an homomorphism of Lie algebra we are ready to introduce the **complete associative algebra** $\tilde{U}(V)$ associated to any vertex algebra V. We will see that for any Lie algebra $\mathfrak{g}$ (not necessarily simple) there is an isomorphism

$$\tilde{U}(V_\kappa(\mathfrak{g})) \simeq \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

Definition 4.2.3. Let V be any vertex algebra and we define $\tilde{U}(V)$ to be the complete associative algebra constructed as follows.

Let $U(U(V))$ the classical enveloping algebra of the Lie algebra $U(V)$ divided by the two sided ideal generated by $1 - (|0\rangle)_{[-1]})$ and consider its completion along the left ideals I_N generated by elements of the form $A_{[n]} : n \geq N$

$$\tilde{U}(U(V)) := \varprojlim U(U(V))/I_N$$

the product may be checked to be continuous under the topology generated by the I_N and therefore $\tilde{U}(U(V))$ is a complete topological associative algebra. Finally define

$$\tilde{U}(V) := \tilde{U}(U(V))/J$$

where J is the two sided ideal generated by the Fourier coefficients of series of the form

$$: Y[A, z]Y[B, z] : -Y[A_{-1}B, z]$$

We have the following proposition

Theorem 4.2.2. *Let $\mathfrak{g}$ be any Lie algebra and let κ be an invariant inner product defined on $\mathfrak{g}$. Consider the affine algebra $\hat{\mathfrak{g}}_\kappa$. Note that all the constructions of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ and $V_\kappa(\mathfrak{g})$ make sense for $\mathfrak{g}$ not necessarily simple.*

Then the homomorphism of Lie algebras introduced above

$$U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

induces an isomorphism

$$\tilde{U}(V_\kappa(\mathfrak{g})) \simeq \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

Proof. Since $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ is an associative algebra we naturally have an homomorphism $U(U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$, which may easily checked to be continuous following the definitions, it therefore induces an homomorphism $\tilde{U}(U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

By proposition 4.2.1 the Fourier coefficients of the series $Y : [A, z]Y[B, z] : -Y[A_{-1}B, z]$ are sent to 0 and therefore we get a well defined map

$$\tilde{U}(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$$

To show that this is an isomorphism we define an inverse. Note that the map

$$\hat{\mathfrak{g}}_\kappa \to \tilde{U}(V_\kappa(\mathfrak{g})) \qquad J^a_n \mapsto (J^a_{-1}|0\rangle)_{[n]} \qquad \mathbf{1} \mapsto (|0\rangle)_{[-1]} = 1$$

is easily checked to be an homomorphism of Lie algebras, indeed

$$\begin{aligned}[(J^a_{-1}|0\rangle)_{[n]}, (J^b_{-1}|0\rangle)_{[m]}] &= \sum_{k\geq 0} \binom{n}{k} (J^a_k J^b_{-1}|0\rangle)_{[n+m-k]} = (J^a_0 J^b_{-1}|0\rangle)_{[n+m]} + n(J^a_1 J^b_{-1}|0\rangle)_{n+m-1} \\ &= ([J^a, J^b]_{-1}|0\rangle)_{[n+m]} + n\kappa(J^a, J^b)(|0\rangle)_{[n+m-1]} = ([J^a, J^b]_{-1}|0\rangle)_{[n+m]} + n\kappa(J^a, J^b)\delta_{n,-m}\end{aligned}$$

□

The last equality follows from the fact that $(|0\rangle)_{[n]} = 0$ for $n \neq -1$, indeed in $V_\kappa(\mathfrak{g}) \otimes \mathbb{C}((t))$ we have $|0\rangle \otimes t^n = \partial\big(\frac{1}{n+1}|0\rangle \otimes t^{n+1}\big)$ for $n \neq -1$.

The homomorphism $\hat{\mathfrak{g}}_\kappa \to \tilde{U}(V_\kappa(\mathfrak{g}))$ induces an homomorphism of associative algebras $U(\hat{\mathfrak{g}}_\kappa) \to \tilde{U}(V_\kappa(\mathfrak{g}))$ which is checked to be continuous and which sends $\mathbf{1} \to 1$ so it induces an homomorphism

$$\widetilde{U}_\kappa(\hat{\mathfrak{g}}) \to \tilde{U}(V_\kappa(\mathfrak{g}))$$

This is easily checked to be the inverse of the homomorphism $\tilde{U}(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})$ indeed it is quite easy to see that both algebras are topologically generated by elements of the form $(J^a_{-1}|0\rangle)_{[n]}$ and J^a_n respectively. By construction both homomorphism send $(J^a_{-1}|0\rangle)_{[n]} \mapsto J^a_n$ and $J^a_n \mapsto (J^a_{-1}|0\rangle)_{[n]}$ and therefore they are one the inverse of the other.

4.3 The center

In the previous section we developed an efficient method to construct central elements in the completed enveloping algebra: taking the vertex operators of central elements of $V_k(\hat{\mathfrak{g}})$. We will therefore focus on the latter center.

The critical value is the only interesting case.

Proposition 4.3.1. *The center $\zeta(V_k(\hat{\mathfrak{g}}))$ is trivial for $k \neq k_c$ (i.e. it is spanned by $|0\rangle$).*

Proof. Consider the normalized Sugawara operators $\tilde{S}$ and a central element $A \in \zeta(V_k(\hat{\mathfrak{g}}))$. By definition of central element we have $S_n A = 0$ for all $n \geq -1$ (recall the shift on the indices in the definition of the Sugawara operators). In particular

$$S_0 A = \deg A = 0$$

Therefore A must be a multiple of $|0\rangle$ since the space $V_k(\hat{\mathfrak{g}})_0$ is one dimensional and spanned by $|0\rangle$. □

We will focus therefore on the critical level $k = k_c$. Since in all the other cases the center is trivial we lighten our notation defining $\zeta(\mathfrak{g}) := \zeta(V_{k_c}(\hat{\mathfrak{g}}))$.

In the case of $V_k(\hat{\mathfrak{g}})$ there is a much more convenient description of the center:

Proposition 4.3.2. *For any $k \in \mathbb{C}$*

$$\zeta(V_k(\hat{\mathfrak{g}})) = V_k(\hat{\mathfrak{g}})^{\mathfrak{g}[[t]]}$$

Proof. The inclusion $\zeta(V_k(\hat{\mathfrak{g}})) \subset V_k(\hat{\mathfrak{g}})^{\mathfrak{g}[[t]]}$ is obvious since by hypothesis every central element S satisfies $J^a_n S = 0$ for all $n \geq 0$. To show the other inclusion consider an invariant element S and consider the centralizer $Z(S) \subset V_k(\hat{\mathfrak{g}})$ which is a vertex subalgebra of $V_k(\hat{\mathfrak{g}})$. Since by hypothesis $Z(S)$ contains all the $J^a_{-1}|0\rangle$ and since the latter elements generate $V_k(\hat{\mathfrak{g}})$ in the sense of the reconstruction theorem we see that it must be

$$Z(S) = V_k(\hat{\mathfrak{g}})$$

So S is actually central. □

This is very nice. To compute $\zeta(V_{k_c}(\mathfrak{g}))$ we will use the approach of graded algebras. The vertex algebra $V_k(\hat{\mathfrak{g}})$ carries a natural filtration induced by the PBW filtration on $U(\hat{\mathfrak{g}}_k)$, the associated graded space is well known and so is its space of invariants this will allow us to put an upper bound to the space $V_{k_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}$.

Let's start with a definition regarding filtered vertex algebras.

Definition 4.3.1. Let V be a vertex algebra. A **filtration** on V is a sequence of subspaces $V_{\leq i}$ for $i \geq 0$ such that:

- $V_{\leq i}$ is a filtration on the vector space V, so $V_{\leq i} \subset V_{\leq i+1}$ and $V = \cup V_{\leq i}$;
- $|0\rangle \in V_{\leq 0}$
- The subspaces $V_{\leq i}$ are T-invariant;
- For $A \in V_{\leq i}$ and $B \in V_{\leq j}$ all the products $A_k B$ for $k \in \mathbb{Z}$ lie in $V_{\leq i+j}$

If V is a filtered vertex algebra the associated graded space (set $V_{\leq -1} = 0$)

$$\operatorname{gr} V := \bigoplus_{i \geq 0} V_{\leq i}/V_{\leq i-1}$$

has a natural structure of a vertex algebra.

Proposition 4.3.3. *The PBW filtration on $V_k(\hat{\mathfrak{g}})$ defines a structure of filtered vertex algebra on $V_k(\hat{\mathfrak{g}})$. The actions of $\operatorname{Der}\mathcal{O}$ and $\mathfrak{g}[[t]]$ are compatible with this filtration (i.e. $\mathfrak{g}[[t]] \cdot V_k(\hat{\mathfrak{g}})_{\leq i} \subset V_k(\hat{\mathfrak{g}})_{\leq i}$).*

Moreover the associated graded vertex algebra is abelian and isomorphic to

$$\operatorname{gr} V_k(\hat{\mathfrak{g}}) \simeq \operatorname{Sym}\left(\frac{\mathfrak{g}((t))}{\mathfrak{g}[[t]]}\right)$$

both as abelian vertex algebras as $\mathfrak{g}[[t]]$ modules and as $\operatorname{Der}\mathcal{O}$ modules. Here the derivation of the space on the right is ∂_t while the structure of $\mathfrak{g}[[t]]$ module is the one induced by the natural action of $\mathfrak{g}[[t]]$ on $\mathfrak{g}((t))/\mathfrak{g}[[t]]$.

Proof. This is a simple verification. □

It is not hard to see that if we denote by $\operatorname{Symb}(A) \in V_{\leq i}/V_{\leq i-1}$ for $A \in V_{\leq i} \setminus V_{\leq i-1}$ that for any $x \in \mathfrak{g}[[t]]$

$$\operatorname{Symb}(x \cdot A) = x \cdot \operatorname{Symb}(A)$$

and therefore

$$\operatorname{gr} \zeta(V_k(\hat{\mathfrak{g}})) \subset \operatorname{Sym}\left(\frac{\mathfrak{g}((t))}{\mathfrak{g}[[t]]}\right)^{\mathfrak{g}[[t]]} \tag{4.2}$$

To study the right hand side of this formula which we denote

$$\operatorname{Inv} \mathfrak{g}^*[[t]] := \operatorname{Sym}\left(\frac{\mathfrak{g}((t))}{\mathfrak{g}[[t]]}\right)^{\mathfrak{g}[[t]]}$$

to justify this notation, but more importantly to compute $\operatorname{Inv}\ \mathfrak{g}^*[[t]]$ we are going need the formalism of Jet schemes.

4.4 Jet Schemes

Definition 4.4.1. Let $X \in \mathbf{Sch}_{\mathbb{C}}$ be any scheme over $\mathbb{C}$. We define functors of $\mathbb{C}$-algebras

$$J_n X(R) := X(R[t]/t^n)$$
$$JX(R) := X(R[[t]])$$

JX is be called the **jet scheme** of X, while $J_n X$ is called the n**-th jet scheme** of X (we will justify this nomenclature with the following proposition). We often denote by X[[t]] the set of $\mathbb{C}$ points of JX:

$$X[[t]] := JX(\mathbb{C}) = X(\mathbb{C}[[t]])$$

To a map $f : X \to Y$ of $\mathbb{C}$-schemes we can associate natural maps $J_n f : J_n X \to J_n Y$ and $Jf : JX \to JY$, these associations are functorial. There are natural maps

$$\pi_{m,n} : J_n X \to J_m X \quad \text{for } n \geq m$$
$$\pi_m : JX \to J_m X$$

The map $JX \to J_n X$ is a cone with respect to the various maps $J_n X \to J_m X$. JX is actually the projective limit of the $J_m X$.

$$JX = \varprojlim J_n X$$

Proposition 4.4.1. *If X is of finite type over $\mathbb{C}$ then $J_n X$ and JX are representable. It turn out that $J_n X$ is of finite type for any n.*

The functoriality of jet schemes is fundamental when talking about algebraic groups and group actions on schemes. Let G be an algebraic group over $\mathbb{C}$. Recall that the groups we start with are always affine and of finite type but we are now going to consider algebraic groups which are not of finite type. Then it is quite clear the $J_n G$ are algebraic groups of finite type and JG is an algebraic group.

If X is a scheme with an action of an algebraic group G then $J_n X$ comes with a natural action of $J_n G$ while JX is equipped with a natural action of JX.

The $\mathbb{C}$ points of the Lie algebras $\mathrm{Lie}(J_n G)$ and $\mathrm{Lie}(JG)$ are given by $\mathfrak{g} \otimes \frac{\mathbb{C}[t]}{t^n} = \mathfrak{g}[t]/t^n$ and $\mathfrak{g} \otimes \mathbb{C}[[t]] = \mathfrak{g}[[t]]$ respectively, both equipped with the bracket induced by the bracket on $\mathfrak{g}$.

Since the spaces $J_n X$ and JX are equipped with a $J_n G$ and a JG action respectively the rings $\mathbb{C}[J_n X]$ and $\mathbb{C}[JX]$ result equipped with an action of $\mathfrak{g}[t]/t^n$ and $\mathfrak{g}[[t]]$ respectively.

As an example consider first the case in which $X = \mathbb{A}^N$ then

$$J_k X(R) = \mathbb{A}^N(R[t]/t^k) = \{(\sum_{n=-1}^{-k} x_{i,n} t^{-n-1})_i : x_{i,n} \in R, i = 1, \dots, N\} \simeq R^{Nk}$$
$$J(X)(R) = \mathbb{A}^N(R[[t]]) = \{(\sum_{n<0} x_{i,n} t^{-n-1})_i : x_{i,n} \in R, i = 1, \dots, N\} \simeq \prod_{n<0} R^N$$

Therefore we have $J_n \mathbb{A}^N \simeq \mathrm{Spec}\, \mathbb{C}[x_{i,n}]_{i=1,\dots,N;-k \leq n<0}$ and $J\mathbb{A}^N \simeq \mathrm{Spec}\, \mathbb{C}[x_{i,n}]_{i=1,\dots,N;n<0}$. This will be our standard notation.

For a morphism:

$$\mathbb{A}^N \xrightarrow{\quad f \quad} \mathbb{A}^M$$

$$\operatorname{Spec}\mathbb{C}[x_1,\dots,x_N] \xrightarrow{(P_1,\dots,P_M)} \operatorname{Spec}\mathbb{C}[y_1,\dots,y_M]$$

Where we read f as $(P_1,\dots,P_m)$: the morphism induced by $y_i \mapsto P_i(x)$. The induced morphism between the jet schemes may be described as follows.

Consider the formal series it t with coefficients in $\mathbb{C}[x_{i,n}]_{i=,1\dots,N,n<0}$ defined by $x_i(t) := \sum_{n<0} x_{i,n} t^{-n-1}$ and the formal substitution

$$P(x) \mapsto P(x(t)) =: \sum_{n<0} P_{i,n}(x) t^{-n-1}$$

where we are using a slight abuse of notation, by $P(x)$ we mean for instance $P(x_1,\dots x_N)$ while by $P_{i,n}(x)$ is a polynomial in the $x_{i,n}$.

Proposition 4.4.2. *The morphism associated to $J(P_1,\dots,P_M)$ between the jet spaces $\operatorname{Spec}\mathbb{C}[x_{i,n}]$ and $\operatorname{Spec}\mathbb{C}[y_{i,n}]$ is the morphism induced by the map of rings $y_{i,n} \mapsto P_{i,n}$ where $P_{i,n}$ is the polynomial defined above.*

Proof. This is just a verification using the definitions. □

We conclude our presentation of jet schemes with a technical but very useful lemma, which we will not prove.

Lemma 4.4.1. *Let $f : X \to Y$ be a morphism between schemes of finite type over $\mathbb{C}$. Suppose that f is formally smooth and surjective (on $\mathbb{C}$ points). Then the morphisms $J_n f : J_n X \to J_n Y$ and $Jf : JX \to JY$ are formally smooth and surjective (on $\mathbb{C}$ points).*

4.4.1 Action of $\operatorname{Aut}\mathcal{O}$ on Jet schemes

Consider a scheme X as always as a functor of $\mathbb{C}$ algebras. There is a natural action of the group $\operatorname{Aut}\mathcal{O}$ on the jet of X:

$$\operatorname{Aut}\mathcal{O}(R) \times JX(R) = \operatorname{Aut}_{cont}(R[[t]]) \times X(R[[t]]) \to X(R[[t]]) \qquad (\rho, x(t)) \mapsto X(\rho)(x)$$

this induces an action of $\operatorname{Aut}\mathcal{O}(\mathbb{C})$ and $\operatorname{Der}\mathcal{O}(\mathbb{C})$ and the algebra of functions on JX.

4.5 Description of $\operatorname{Inv}\mathfrak{g}^*[[t]]$

We are now ready to give a complete description of the space $\operatorname{Inv}\mathfrak{g}^*[[t]] = \operatorname{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)^{\mathfrak{g}[[t]]}$ but first we have to reinterpret the algebra $\operatorname{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)$ as the algebra of functions on a geometric space, here jet schemes will come in the picture.

Let $\mathfrak{g}^*$ be the scheme associated to the dual vector space of the Lie algebra $\mathfrak{g}$. Pick a basis J^a of $\mathfrak{g}$ and denote by $\overline{J^a}$ the linear functional on $\mathfrak{g}^*$ defined by evaluating $\varphi \in \mathfrak{g}^*$ on the J^a. So

$$\overline{J^a}(\varphi) := \varphi(J^a)$$

It is clear that the elements $\overline{J^a}$ form basis of $\mathfrak{g}^{**}$ and that

$$\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[\overline{J^a}]$$

With the notations we used so far we have that $\mathbb{C}[J\mathfrak{g}^*] = \mathbb{C}[\overline{J^a}_n]_{n<0}$, furthermore notice that $\mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right) = \mathbb{C}[J^a_n]_{n<0}$. This notation is more than a mere coincidence.

Indeed consider the linear map

$$\mathfrak{g}((t))/\mathfrak{g}[[t]] \to \left(\mathfrak{g}^*[[t]]\right)^* \qquad \xi(t) \mapsto \left(\varphi(t) \mapsto \int \langle \varphi(t), \xi(t) \rangle dt\right)$$

which identifies for instance J^a_n with $n < 0$ with the linear functional on $\mathfrak{g}^*[[t]]$ defined by $\sum_{n<0} \varphi_n t^{-n-1} \mapsto \varphi_n(J^a)$. This is a linear continuous functional on the vector space $\mathfrak{g}^*[[t]]$, which consists in exactly the $\mathbb{C}$ points of the scheme $J\mathfrak{g}^*$.

These descriptions suggest the following proposition.

Proposition 4.5.1. *The map*

$$\mathbb{C}[J\mathfrak{g}^*] \to \mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right) \qquad \overline{J^a}_n \mapsto J^a_n$$

is an isomorphism of algebras and of $\mathfrak{g}[[t]]$ *and* $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$ *modules. Where the action of* $\mathfrak{g}[[t]]$ *on* $\mathbb{C}[J\mathfrak{g}^*]$ *is induced through the jet functor as described before by the coadjoint action of* G *on* $\mathfrak{g}^*$*, while the action of* $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$ *is the one induced by the action of* $\mathrm{Aut}\,\mathcal{O}$ *on* $J\mathfrak{g}^*$*.*

Proof. The fact that it is an isomorphism of algebra is clear from the above descriptions. To see that it commutes with the action $\mathfrak{g}[[t]]$ notice that since G acts on $\mathfrak{g}^*$ with the coadjoint action $\mathfrak{g}$ acts on $\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g}$ through the adjoint action. From these premises, using the definition of jet schemes and of the induced action it is not difficult to conclude the the isomorphism is $\mathfrak{g}[[t]]$ equivariant. An analogous verification proves $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$ invariancy as well. □

Corollary 4.5.1. *The isomorphism* $\mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right) \simeq \mathbb{C}[J\mathfrak{g}^*]$ *induces an isomorphism on the space of invariants*

$$\mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]} \simeq \mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)^{\mathfrak{g}[[t]]} = \mathrm{Inv}\,\mathfrak{g}^*[[t]]$$

This also justifies the notation $\mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)^{\mathfrak{g}[[t]]} = \mathrm{Inv}\,\mathfrak{g}^*[[t]]$.

Now that we described or ring of invariants in a geometric way we can give the desired description of $\mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)^{\mathfrak{g}[[t]]}$. We consider first the finite dimensional case.

As we saw in the preliminaries the space of polynomial invariants of $\mathbb{C}[\mathfrak{g}^*]$ is a free polynomial algebra, generated by $l = \dim\mathfrak{h}$ homogeneous polynomials $\overline{P}_1, \dots, \overline{P}_l$:

$$\mathbb{C}[\mathfrak{g}^*]^{G(\mathbb{C})} = \mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}(\mathbb{C})} = \mathbb{C}[\overline{P}_1, \dots, \overline{P}_l] =: \mathrm{Inv}\,\mathfrak{g}^*$$

s There is more, consider $\mathcal{P} := \mathrm{Spec}\,\mathbb{C}[\overline{P}_i]$ and consider the morphism induced by the inclusion $\mathbb{C}[\overline{P}_i] \hookrightarrow \mathbb{C}[\mathfrak{g}^*]$

$$p : \mathfrak{g}^* \to \mathcal{P}$$

The following theorem is due to Kostant [Kos63].

Theorem 4.5.1. *Let $\mathfrak{g}^*_{reg}$ be the open subscheme of $\mathfrak{g}^*$ defined by*

$$\mathfrak{g}^*_{reg} := \{x \in \mathfrak{g}^* : \dim \mathfrak{g}_x = l\}$$

then the map obtained by restriction

$$p : \mathfrak{g}^*_{reg} \to \mathcal{P}$$

is smooth (and hence formally smooth), surjective and its fibers are single G orbits. So $p : \mathfrak{g}^*_{reg} \to \mathcal{P}$ is a geometric quotient. In particular

$$\mathbb{C}[\mathfrak{g}^*_{reg}]^{\mathfrak{g}(\mathbb{C})} = p^\#\mathbb{C}[\mathcal{P}]$$

Using the formalism of jet schemes we are now ready to pass from the finite dimensional case to the computation of $\mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$.

Theorem 4.5.2. *The map induced by $Jp : J\mathfrak{g}^* \to J\mathcal{P}$ on the ring of functions*

$$Jp^\# : \mathbb{C}[J\mathcal{P}] \to \mathbb{C}[J\mathfrak{g}^*]$$

is injective and induces an isomorphism.

$$\mathbb{C}[J\mathcal{P}] = \mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$$

In particular if we take $\overline{P_i}$ as the polynomials in $\mathbb{C}[\mathfrak{g}^*]$ which generate as above the free subalgebra $\mathbb{C}[\mathcal{P}]$ then we have

$$\mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]} = \mathbb{C}[\overline{P}_{i,n}]_{i=1,\dots,l;n<0}$$

where the $P_{i,n}$ are defined as in proposition 4.4.2.

Proof. We will prove first the 'finite dimensional' case.

Lemma 4.5.1. *For every $n \geq 0$ the map induced by $J_n p : J_n\mathfrak{g}^*_{reg} \to J_n\mathcal{P}$ is a geometric quotient with respect to the action of $J_n G$.*

In particular since all schemes appearing are of finite type

$$\mathbb{C}[J_n\mathfrak{g}^*_{reg}]^{\mathfrak{g}[t]/t^n} = \mathbb{C}[J_n\mathcal{P}]$$

where we identify $\mathbb{C}[J_n\mathcal{P}]$ with its image under $J_n p^\#$ which is injective.

Indeed since $p : \mathfrak{g}^*_{reg} \to \mathcal{P}$ is smooth by theorem 4.5.1 we obtain applying lemma 4.4.1 that $J_n\mathfrak{g}^*_{reg} \to J_n\mathcal{P}$ is smooth and surjective. For the same reasons the map

$$J_n G \times J_n\mathfrak{g}^*_{reg} \to J_n\mathfrak{g}^*_{reg} \times_{J_n\mathcal{P}} J_n\mathfrak{g}^*_{reg}$$

is surjective, therefore the geometric fibers of $J_n p$ consist in single $J_n G$ orbits and we may apply theorem 2.2.2 to find out that $J_n p : J_n\mathfrak{g}^*_{reg} \to J_n\mathcal{P}$ is again a geometric quotient.

Corollary 4.5.2. *The injective map*

$$J_n p^\# : \mathbb{C}[J_n\mathcal{P}] \to \mathbb{C}[J_n\mathfrak{g}^*]$$

induces an isomorphism

$$\mathbb{C}[J_n\mathcal{P}] = \mathbb{C}[J_n\mathfrak{g}^*]^{\mathfrak{g}[t]/t^n}$$

Since the composition $\mathbb{C}[J_n\mathcal{P}] \to \mathbb{C}[J_n\mathfrak{g}^*] \to \mathbb{C}[J_n\mathfrak{g}^*_{reg}]$ is injective the first map must be injective as well, in addition all functions coming from $J_n\mathcal{P}$ are automatically J_nG invariant and therefore $\mathfrak{g}[t]/t^n$ invariant as well. So

$$\mathbb{C}[J_n\mathcal{P}] \subset \mathbb{C}[J_n\mathfrak{g}^*]^{\mathfrak{g}[t]/t^n}$$

To show the other inclusion consider a $\mathfrak{g}[t]/t^n$ invariant function on $J_n\mathfrak{g}^*$. Its restriction to $J_n\mathfrak{g}^*_{reg}$ which is an open subscheme of $J_n\mathfrak{g}^*$ is still $\mathfrak{g}[t]/t^n$ invariant and by the lemma it belongs to $\mathbb{C}[J_n\mathcal{P}]$.

We are now ready to prove the statement of the theorem. Note first the following remark, which follows from the description we gave of the ring of functions on jet schemes.

Remark 4.5.1. Any regular function $f : J\mathbb{A}^N \to \mathbb{A}^1$ comes from a regular function $\tilde{f} : J_n\mathbb{A}^N \to \mathbb{A}^1$ under the injective morphism $\pi_n^\#$, induced by the natural map $\pi_n : J\mathbb{A}^N \to J_n\mathbb{A}^N$.

The map $Jp^\#$ is injective. Indeed consider a regular function $f \in \mathbb{C}[J\mathcal{P}]$ which is sent to $0 \in \mathbb{C}[J\mathfrak{g}^*]$ under $Jp^\#$. By the above remark there exists a positive integer n such that f comes from a function $\tilde{f} : J_n\mathcal{P} \to \mathbb{A}^1$. Since the following diagram is commutative

$$\begin{array}{ccc} J\mathfrak{g}^* & \xrightarrow{\pi_n} & J_n\mathfrak{g}^* \\ {\scriptstyle Jp}\downarrow & & \downarrow{\scriptstyle J_n p} \\ J\mathcal{P} & \xrightarrow{\pi_n} & J_n\mathcal{P} \end{array}$$

and since $J_np^\#$ is injective we must have $\tilde{f} = 0$ and hence $f = 0$.

In addition any function which is the pullback of a function on $J\mathcal{P}$ is automatically G invariant. Therefore we have

$$\mathbb{C}[J\mathcal{P}] \subset \mathbb{C}[J\mathfrak{g}^*]^{JG} \subset \mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$$

To show the other inclusion consider a function $f \in \mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$ and pick $\tilde{f} : J_n\mathbb{C}[J_n\mathfrak{g}^*] \to \mathbb{A}^1$ such that $f = \pi_n^\# \tilde{f}$. We would like to show that $\tilde{f}$ is $\mathfrak{g}[t]/t^n$ invariant. Consider the commutative diagram

$$\begin{array}{ccccccccc} J\mathfrak{g} \times J\mathfrak{g}^* & \longrightarrow & T(JG \times J\mathfrak{g}^*) & \xrightarrow{T(J\mu)} & T(J\mathfrak{g}^*) & \xrightarrow{Tf} & T\mathbb{A}^1 & \xrightarrow{\frac{d}{d\epsilon}} & \mathbb{A}^1 \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ J_n\mathfrak{g} \times J_n\mathfrak{g}^* & \longrightarrow & T(J_nG \times J_n\mathfrak{g}^*) & \xrightarrow[T(J_n\mu)]{} & T(J_n\mathfrak{g}^*) & \xrightarrow[T\tilde{f}]{} & T\mathbb{A}^1 & \xrightarrow[\frac{d}{d\epsilon}]{} & \mathbb{A}^1 \end{array}$$

Notice that the map $J\mathfrak{g} \times J\mathfrak{g}^* \to J_n\mathfrak{g} \times J_n\mathfrak{g}^*$ is surjective on $\mathbb{C}$ points. Therefore given an element $(\xi, x) \in J_n\mathfrak{g} \times J_n\mathfrak{g}^*$ we may pick $(\Xi, X) \in J\mathfrak{g} \times J\mathfrak{g}^*$ which maps to (ξ, x), by hypothesis the upper row, evaluated on (Ξ, X) is equal to 0 and by commutativity the lower row must be 0 evaluated on (ξ, x).

This implies that $\tilde{f}$ is $\mathfrak{g}[t]/t^n$ invariant, since we proved that on $\mathbb{C}$ points $\xi \cdot \tilde{f} : J_n\mathfrak{g}^*(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$ is 0 and $J_n\mathfrak{g}^*$ is of finite type. Now since $\tilde{f}$ is $\mathfrak{g}[t]/t^n$ invariant, by the corollary above must belong to $\mathbb{C}[J_n\mathcal{P}]$ and consequently we have $f \in \mathbb{C}[J_n\mathcal{P}] \subset \mathbb{C}[J\mathcal{P}]$. □

We conclude this section calculating the character of $\mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$ under the action of $L_0 = -t\partial_t$. Recall that this action intertwines with the isomorphism

$$\mathbb{C}[J\mathfrak{g}^*] \simeq \mathrm{Sym}\left(\mathfrak{g}((t))/\mathfrak{g}[[t]]\right)$$

In particular we have $L_0(\overline{J^a}_n) = n\overline{J^a}_n$. Starting from this it is quite easy to compute the character of the $P_{i,n}$. P_i is an homogeneous polynomial of degree $d_i + 1$. Therefore the n-th coefficient of

$$P(t) = P(J^a(t)) = \sum_{n<0} P_{i,n} t^{-n-1}$$

(i.e. $P_{i,n}$) will be a finite sum of products of the form $J^{a_1}_{n_1} \dots J^{a_{d_i}}_{n_{d_i}}$ with $\sum n_j = -n + d_i$.

Proposition 4.5.2. *The character of* $\mathrm{Inv}\ \mathfrak{g}^*[[t]]$ *is given by*

$$\mathrm{ch}(\mathrm{Inv}\ \mathfrak{g}^*[[t]]) = \prod_{i=1}^{l} \prod_{n_i \geq d_i+1} \frac{1}{(1-q^{n_i})}$$

4.6 $\zeta(\mathfrak{g})$ and $Z_\kappa(\hat{\mathfrak{g}})$

The following theorem will be our goal for the rest of the thesis, therefore we will postpone its proof, as the definition of the space of Opers, until the end of the thesis.

Theorem 4.6.1. *The center of the vertex algebra* $\zeta(\mathfrak{g})$ *is isomorphic in a* $(\mathrm{Aut}\,\mathcal{O}, \mathrm{Der}\,\mathcal{O})$ *equivariant way to the algebra of regular functions on the space of* ${}^LG$ *Opers on the disc:* $\mathrm{Op}_{{}^LG}(D)$.

Its character is given by the formula

$$\mathrm{ch}(\zeta(\mathfrak{g})) = \prod_{i=1}^{l} \prod_{n_i \geq d_i+1} \frac{1}{(1-q^{n_i})}$$

We will see that actually the two statements are actually separate and that that the second one implies the first. We state them in one theorem here just for convenience.

In particular the proof of the second statement follows from the fact that $\mathrm{gr}\,(\zeta(\mathfrak{g})) = \mathrm{Inv}\ \mathfrak{g}^*[[t]]$ this is the fact that we are actually use the most.

The algebra of functions on the space of Opers is a free polynomial algebra in the variables $v_{i,n}$

$$\zeta(\mathfrak{g}) = \mathbb{C}[v_{i,n}]_{i=1,\dots,l;n<0}$$

To proceed with our description of the center of the completed enveloping algebra we need first a some little geometric definitions.

4.6.1 Loop schemes

Definition 4.6.1. Given a $\mathbb{C}$ scheme X (or a functor of $\mathbb{C}$-algebras) we define LX as the functor of $\mathbb{C}$ algebras

$$LX(R) := X(R((t)))$$

while to a morphism $f : R_1 \to R_2$ we first associate $Lf : R_1((t)) \to R_2((t))$ which is naturally defined as $\sum_i r_i t^{-i-1} \mapsto \sum_i f(r_i) t^{-i-1}$, and then we associate the induced morphism $X(R_1((t))) \to X(R_2((t))$. Notice that LX carries an action of $\mathrm{Aut}\,\mathcal{O}$ in the same way JX does.

Given any scheme X the functor LX is rarely a scheme, but in the situations we are interested in it will often be an ind-scheme.

We consider the case we are most interested with, which is also the simplest one: $X = \mathbb{A}^m$.

Define for $N \geq 0$ subfunctors

$$L_N\mathbb{A}^m(R) := \left\{(x_1(t), \ldots, x_m(t) \in \mathbb{A}^m(R((t))) : x_i(t) \in t^{-N}R[[t]]\right\}$$

It is easy to see that

$$L\mathbb{A}^m = \varinjlim L_N\mathbb{A}^m$$

Denote by $x_{i,n}$ the regular function on $L\mathbb{A}^m$ (which we recall to be defined as a natural transformation $L\mathbb{A}^m \to \mathbb{A}^1$) which on R points

$$L\mathbb{A}^m(R) \ni (\sum_j r_{1,j}t^{-j-1}, \ldots, \sum_j r_{m,j}t^{-j-1}) \mapsto r_{i,n} \in \mathbb{A}^1(R)$$

The ring of regular functions on $L\mathbb{A}^n$, is easily seen to be described as the projective limit of the rings $\mathbb{C}[L_N\mathbb{A}^m]$ which is isomorphic to

$$\mathbb{C}[L\mathbb{A}^m] = \varprojlim \frac{\mathbb{C}[x_{i,n}]_{i=1,\ldots,m;n\in\mathbb{Z}}}{(x_{i,n})_{n\geq N}}$$

Moreover is quite clear that the map $L_N\mathbb{A}^m(R) \to J\mathbb{A}^m(R)$ which sends

$$(\sum_{n<N} r_{i,n}t^{-n-1})_i \mapsto (\sum_{n<0} r_{i,n+N}t^{-n-1})_i$$

is a functorial isomorphism. So $L_N\mathbb{A}^m$ is an affine scheme for every N which is isomorphic to

$$\operatorname{Spec} \mathbb{C}[x_{i,n}]_{i=1,\ldots,m;n\leq N-1}$$

Consider now a polynomial function $P = P(x_1, \ldots, x_m) : \mathbb{A}^m \to \mathbb{A}^1$. We want to describe the induced regular function $LP : L\mathbb{A}^m \to L\mathbb{A}^1$ in particular its composition with the coordinate function $x_n : L\mathbb{A}^1 \to \mathbb{A}^1$ (defined as before) which we call P_n.

Consider our usual notation $x_i(t) := \sum_{n\in\mathbb{Z}} x_{i,n}t^{-n-1}$ and let

$$P(t) := P(x_1(t), \ldots, x_m(t)) = \sum_{n\in\mathbb{Z}} P_n t^{-n-1}$$

Note that here there is a slight ambiguity since in the 'definition' of the P_n infinite sums occur. The following lemma resolves this ambiguity.

Lemma 4.6.1. *The P_n above are well defined as elements of the completed algebra generated by the variables $x_{i,m}$*

$$P_n \in \varprojlim \frac{\mathbb{C}[x_{i,n}]_{i=1,\ldots,m;n\in\mathbb{Z}}}{(x_{i,n})_{n\geq N}}$$

therefore P_n is a well function in $\mathbb{C}[L\mathbb{A}^m]$ and it corresponds exactly to the composition $x_n \circ LP$.

Proof. These are just verifications that directly follow from the definitions. □

We are now going to investigate what happens when we restrict the functions P_n on the sub-functors $L_N\mathbb{A}^m$.

Remark 4.6.1. When we restrict P_n to the subspace $L_N\mathbb{A}^m$ all the functions x_n with $n \geq N$ vanish (since they are 0 on that space). So in the computation of the P_n we can just make the substitution $x_i \mapsto x_{i,N}(t) = \sum_{n<N} x_i t^{-n-1}$ and compute the coefficients of $P(t) = P(x_{1,N}(t), \ldots, x_{m,N}(t))$.

Note that when $N > 0$ and P is homogeneous of degree d the only P_n which we know for sure that are identically 0 are those for $n > Nd$, even if other cancellations may occur a priori.

4.6.2 Back to the center

We turn now to our case of interest: the one where $X = \mathfrak{g}^*$. Note that $\mathfrak{g}^* \simeq \mathbb{A}^{\dim \mathfrak{g}}$ as a $\mathbb{C}$ scheme so all we have done so far applies to this case as well. We group what we will need in what follows in the following lemma.

Lemma 4.6.2. *The group* JG *acts in a natural way on all spaces* $L_N\mathfrak{g}^*$ *and* $L\mathfrak{g}$*. In addition the isomorphisms introduced above*

$$\cdot t^n : L_N\mathfrak{g}^* \to J\mathfrak{g}^*$$

intertwine with this action. In particular the induced isomorphisms on the ring of functions intertwine with the action of $JG(\mathbb{C}) = G[[t]]$ *and of* $\mathfrak{g}[[t]]$.

Let $\overline{P}_i \in \mathbb{C}[\mathfrak{g}^*]$ *be the free generators of the algebra of invariant functions. And let the* $P_{i,n}$ *be the regular functions on* $L\mathfrak{g}^*$ *defined above. We have*

$$\mathbb{C}[L_N\mathfrak{g}^*]^{\mathfrak{g}[[t]]} = \mathbb{C}[P_{i,n_i}]_{n_i \leq N(d_i+1)}$$

and this is a free polynomial algebra. Here the P_{i,n_i} *are the functions defined above restricted to* $L_N\mathfrak{g}^*$.

Proof. The fact that JG acts on $L_N\mathfrak{g}^*$ comes from the fact that LG acts on $L\mathfrak{g}^*$ and JG is a subgroup of which preserves the space $L_N\mathfrak{g}^*$. This last affirmation may be proved for instance viewing G as a matrix group. The fact that the isomorphism above intertwines with the action of G may be proved viewing G as a matrix group as well.

The last properties follows from the fact that the isomorphism $\cdot t^N$ intertwines with the action of JG and from the fact that under the isomorphism above $P_{i,n}$ is sent to $P_{i,n+Nd_i}$ □

We are now ready to find out what is the relation of these algebras of functions with the enveloping algebra. Recall that by definition of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ we have

$$\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N = \big(U(\hat{\mathfrak{g}}_k)/(1-\mathbf{1})\big)/I_N$$

where in both cases I_N is the left ideal generated by $t^N\mathfrak{g}[[t]]$.

Lemma 4.6.3. *The PBW filtration on* $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ *induces a filtration on* $\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N$*, the associated graded space* $\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N)$ *has a natural structure of commutative* $\mathbb{C}$ *algebra which comes from the structure of algebra on* $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$.

In addition there is an isomorphism of $\mathbb{C}$ *algebras*

$$\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N) \simeq \mathbb{C}[L_N\mathfrak{g}^*]$$

which intertwines with the action of $\mathfrak{g}[[t]]$ *on both spaces.*

Proof. Consider the surjective morphism

$$\mathrm{gr}\,(U(\hat{\mathfrak{g}}_k)) \to \mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N)$$

which arises from the quotient morphism $U(\hat{\mathfrak{g}}_k) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N$ which preserves the PBW filtration on both spaces. Surjectivity follows from the fact that the PBW filtration on the quotient is defined exactly as the image of the filtration of the above morphism.

Is not difficult to see that $\mathrm{gr}\,(U(\hat{\mathfrak{g}}_k)) \simeq \mathrm{Sym}\,\mathfrak{g}((t))$ as an algebra: the commutation relations of element in $\hat{\mathfrak{g}}_k$ differ from the ones in $\mathfrak{g}((t))$ but the additional terms produce terms with lower degree and a proof analogous to the one of the classical PBW theorem proves the assertion.

The kernel of the above map is easily seen to be the ideal generated by $t^N\mathfrak{g}[[t]]$, so $\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N)$ has a natural structure of commutative algebra and it is isomorphic to

$$\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N) \simeq \frac{\mathrm{Sym}\,\mathfrak{g}((t))}{(t^N\mathfrak{g}[[t]])}$$

Finally, similarly to proposition 4.5.1 the latter space is isomorphic, in a $\mathfrak{g}[[t]]$ equivariant way, to $\mathbb{C}[L_N\mathfrak{g}^*]$. ☐

Next consider the isomorphism presented at the beginning of the section

$$\mathrm{gr}\,(\zeta(\mathfrak{g})) = \mathrm{Inv}\;\mathfrak{g}^*[[t]] = \mathbb{C}[\overline{P}_{i,n}]_{i=1,\dots,l;n<0}$$

and pick elements $S_i \in \zeta(\mathfrak{g})$ such that $\mathrm{Symb}\,S_i = \overline{P_{i,-1}}$. This means that S_i, up to an element of $V_{k_c}(\mathfrak{g})_{\leq 1}$, is equal to the polynomial P_i where we make the formal substitution $J^a \mapsto J^{-1}$ applied to $|0\rangle$. This leads to the following fact.

Lemma 4.6.4. *For any $i = 1,\dots,l$ any $N > 0$ and any $n \in \mathbb{Z}$ the image of*

$$\Phi((S_i)_{[n]}) \in \widetilde{U}_\kappa(\hat{\mathfrak{g}}) \to \widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N$$

has symbol in $\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N) = \mathbb{C}[L_N\mathfrak{g}^*]$ equal to $P_{i,n} \in \mathbb{C}[L_N\mathfrak{g}^*]$.

Proof. This follows from the definitions. Recall that $\Phi((S_i)_{[n]})$ was defined as the n-th coefficient of the sum of normally ordered products defined by $Y[S_i, z]$. Considering its symbol in $\mathrm{gr}\,(\widetilde{U}_\kappa(\hat{\mathfrak{g}})/I_N)$ we may forget about the ordering (as the latter algebra is commutative) and put all terms of the form J^a_n with $n \geq N$ equal to 0. Eliminating the lower degree term we find out that the expression we obtain is exactly equal to the one of the polynomials $P_{i,n}$ which was obtained with the formal substitution $J^a \mapsto J^a(t) = \sum_{n<N} J^a_n t^{-n-1}$.

This concludes the proof. ☐

Proposition 4.6.1. *Let I_N be the left ideal in $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ generated by $t^N\mathfrak{g}[[t]]$. Then the quotients of the center by these ideals (which restricted to $Z_\kappa(\hat{\mathfrak{g}})$ are bilateral) is given by*

$$\frac{Z_\kappa(\hat{\mathfrak{g}})}{Z_\kappa(\hat{\mathfrak{g}}) \cap I_N} \simeq \mathbb{C}[S_{i,[n]}]_{i=1,\dots,l;N\leq N(d_i+1)}$$

in addition, this is a free polynomial algebra.

Proof. Note that $Z_{k_c}(\hat{\mathfrak{g}}) \subset \tilde{U}_{k_c}(\hat{\mathfrak{g}})^{\mathfrak{g}[[t]]}$. Since the operation of taking the symbol commutes with the action of $\mathfrak{g}[[t]]$ on the spaces $\tilde{U}_{k_c}(\hat{\mathfrak{g}})/I_N$ and $\operatorname{gr}\left(\tilde{U}_{k_c}(\hat{\mathfrak{g}})/I_N\right)$. We have the following chain of inclusions:

$$\operatorname{gr}\left(\frac{Z_{k_c}(\hat{\mathfrak{g}})}{Z_{k_c}(\hat{\mathfrak{g}}) \cap I_N}\right) \subset \operatorname{gr}\left(\left(\frac{\tilde{U}_{k_c}(\hat{\mathfrak{g}})}{I_N}\right)^{\mathfrak{g}[[t]]}\right) \subset \operatorname{gr}\left(\frac{\tilde{U}_{k_c}(\hat{\mathfrak{g}})}{I_N}\right)^{\mathfrak{g}[[t]]} = \mathbb{C}[P_{i,n_i}]_{i=1,\dots,l;n_i \leq N(d_i+1)}$$

We know, from lemma 4.6.4 that for each $P_{i,n}$ there is a central element, namely $\Phi(S_{i,[n]})$, whose symbol is exactly $P_{i,n}$. All the above inclusions are therefore equalities and

$$\operatorname{gr}\left(\frac{Z_{k_c}(\hat{\mathfrak{g}})}{Z_{k_c}(\hat{\mathfrak{g}}) \cap I_N}\right) \simeq \mathbb{C}[P_{i,n_i}]_{i=1,\dots,l;n_i \leq N(d_i+1)}$$

The thesis follows from the following general fact.

Lemma 4.6.5. *Let A be a commutative $\mathbb{C}$ algebra. Suppose A carries a filtration such that the associated graded algebra is a free polynomial algebra.*

$$\operatorname{gr} A \simeq \mathbb{C}[x_i]_{i \in I}$$

Then, taken any $a_i \in A$ such that $\operatorname{Symb} a_i = x_i$, A is a free polynomial algebra generated by the a_i.

□

This proposition leads to this first description of the center of the enveloping algebra.

Corollary 4.6.1. *Consider $S_{i,[n]} \in Z_{k_c}(\hat{\mathfrak{g}})$ as above. Then the $S_{i,[n]}$ are algebraically independent. Moreover $Z_{k_c}(\hat{\mathfrak{g}})$ is the completion of its free polynomial subalgebra $\mathbb{C}[S_{i,[n]}]_{i=1,\dots,l;n\in\mathbb{Z}}$ by the ideals*

$$(S_{i,[n_i]})_{i=1,\dots,l;n_i > N(d_i+1)}$$

Proof. Any algebraic relation between the $S_{i,[n_i]}$ must contain a finite number of terms. Therefore we may consider an N sufficiently large such that all the terms appearing are of the form $S_{i,[n_i]}$ with $n_i \leq N d_i$. Consider this algebraic relation to the quotient

$$\frac{Z_{k_c}(\hat{\mathfrak{g}})}{Z_{k_c}(\hat{\mathfrak{g}}) \cap I_N}$$

we see that the algebraic expression itself (i.e. the polynomial in the $S_{i,[n_i]}$) must be 0, since the quotient is the free polynomial algebra described above. This proves that the elements $S_{i,[n]}$ are algebraically independent.

The second statement easily follows from the fact that $Z_{k_c}(\hat{\mathfrak{g}})$ is the projective limit of the quotients

$$\frac{Z_{k_c}(\hat{\mathfrak{g}})}{Z_{k_c}(\hat{\mathfrak{g}}) \cap I_N}$$

which are isomorphic to the subalgebra of the $S_{i,[n]}$ modulo the ideal generated by

$$(S_{i,[n_i]})_{i=1,\dots,l;n_i > N(d_i+1)}$$

□

Corollary 4.6.2. *The homomorphism*

$$U(\zeta(\mathfrak{g})) \to Z_{\kappa_c}(\hat{\mathfrak{g}})$$

induces an isomorphism

$$\tilde{U}(\zeta(\mathfrak{g})) \simeq Z_{\kappa_c}(\hat{\mathfrak{g}})$$

Proof. Consider the $\mathrm{Der}\,\mathcal{O}$ equivariant isomorphism $\mathrm{gr}\,(\zeta(\mathfrak{g})) = \mathrm{Inv}\;\mathfrak{g}^*[[t]]$ and as before let $S_i \in \zeta(\mathfrak{g})$ be elements such that $\mathrm{Symb}\,S_i = P_{i,-1}$ it follows by construction that $\mathrm{Symb}\,S_{i,n}|0\rangle = P_{i,n}$ for every $n < 0$ and therefore

$$\zeta(\mathfrak{g}) = \mathbb{C}[S_{i,n}|0\rangle]_{i=1,\dots,l,n<0}$$

as a commutative algebra. In particular $\zeta(\mathfrak{g}) \simeq V_0(S)$ for $S = \oplus_i \mathbb{C} S_i$ an abelian algebra of dimension l. Now by Theorem 4.2.2 we have

$$\tilde{U}(\zeta(\mathfrak{g})) \simeq \varprojlim \frac{\mathbb{C}[S_{i,n}]_{i=1,\dots,l,n\in\mathbb{Z}}}{(S_{i,n})_{n>N}}$$

following the definitions and by the description we gave above of the center as the completion of its subalgebra generated by the $S_{i,[n]}$ it easily follows that the induced map

$$\tilde{U}(\zeta(\mathfrak{g})) \to Z_{\kappa_c}(\hat{\mathfrak{g}})$$

is an isomorphism. □

Finally we state the following lemma, which can be found in [Fre07, Lemma 4.3.5]

Lemma 4.6.6. *The completed algebra* $\tilde{U}(\mathbb{C}[\mathrm{Op}_{{}^LG}(D)]$ *is isomorphic in a* $(\mathrm{Aut}\,\mathcal{O}, \mathrm{Der}\,\mathcal{O})$ *equivariant way to the algebra of regular functions on the space of* ${}^L G$*-Opers on the pointed disc.*

$$\tilde{U}(\mathbb{C}[\mathrm{Op}_{{}^LG}(D)) = \mathbb{C}[\mathrm{Op}_{{}^LG}(D^*)]$$

From which easily follows the description of the center of the completed enveloping algebra we wanted to prove.

Theorem 4.6.2. *The center of the completed enveloping algebra at the critical level* $Z_{\kappa_c}(\hat{\mathfrak{g}})$ *is isomorphic in a* $(\mathrm{Aut}\,\mathcal{O}, \mathrm{Der}\,\mathcal{O})$ *equivariant way to the algebra of functions on the space of* ${}^L G$*-Opers on the pointed disc*

$$Z_{\kappa_c}(\hat{\mathfrak{g}}) = \mathbb{C}[\mathrm{Op}_{{}^LG}(D^*)]$$

Chapter 5

Free field realization

In this chapter we begin with the proof of the description of the center of the vertex algebra $V_{\kappa_c}(\mathfrak{g})$. A pivotal role in the proof will be the construction of certain $\hat{\mathfrak{g}}_{\kappa_c}$ modules, called the **Wakimoto modules**, they were first defined by Wakimoto [Wak86] for the algebra $\widehat{\mathfrak{sl}}_{2\,\kappa_c}$ and then generalized to an arbitrary simple algebra $\mathfrak{g}$ by Feigin and Frenkel [FF88].

The exposition, as well as the theorems themselves, will be carried out using the vertex algebra language. In order to define the Wakimoto modules we need to construct what is called a **free field realization** of the vertex algebra $V_\kappa(\mathfrak{g})$ that is to say a vertex algebra homomorphism

$$V_\kappa(\mathfrak{g}) \to V$$

where V is a 'free field' algebra: a vertex algebra which is generated (in the sense of the reconstruction theorem) by fields $a_\alpha(z)$ whose commutator is constant (i.e. its not a function of other non constant fields). In this chapter we describe in detail the free field realization, while in the next chapter we will describe Wakimoto modules and their properties.

5.1 The finite dimensional case

We start with our usual simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$ and let $\mathfrak{b}_+$ and $\mathfrak{b}_-$ the upper and lower Borel subalgebras induced by a choice of a basis Δ_+ for the set of roots Φ. Let $\mathfrak{n}_+$ and $\mathfrak{n}_-$ be the upper and lower Borel subalgebras: $\mathfrak{n}_\pm = [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$.

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

We denote by $(e_\alpha)_{\alpha\in\Phi}$ ($(f_\alpha)_{\alpha\in\Phi}$ resp.) a standard basis of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) such that $[h, e_\alpha] = \alpha(h)e_\alpha$ (resp. $[h, f_\alpha] = -\alpha(h)f_\alpha$ for any $h \in \mathfrak{h}$.

Let G be the connected simply connected Lie group associated to $\mathfrak{g}$. To the above decomposition of $\mathfrak{g}$ are associated various subgroups of G: the maximal toral subgroup H of G, the upper Borel subgroup $H \subset B_+ \subset G$ and the lower Borel subgroup $H \subset B_- \subset G$. Their Lie algebras are respectively $\mathfrak{h}, \mathfrak{b}_-, \mathfrak{b}_+$. Finally let N_+ and N_- be the unipotent subgroups of G associated to the nilpotent Lie subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$.

We consider the flag variety G/B_-, it has a unique open B_+ orbit

$$U := N_+ \cdot [1] \subset G/B_-$$

The left multiplication by elements of G induces an action

$$G \times G/B_- \to G/B_-$$

which induces an action by vector fields

$$\mathfrak{g} \to \mathrm{Vect}(U) = \mathrm{Der}\,\mathbb{C}[N_+]$$

We will keep in mind that $U \simeq N_+$, freely interchanging them, we list here some useful remarks.

Remark 5.1.1. Identifying $U = N_+$ the action of $\mathfrak{g}$ on $\mathrm{Vect}(N_+)$ satisfy the following properties:

- The action of $\mathfrak{n}_+$ coincides with the action of $\mathfrak{n}_+$ on the ring of functions $\mathbb{C}[N_+]$ induced by the left multiplication
$$N_+ \times N_+ \to N_+$$
- The action of $\mathfrak{h}$ corresponds to the action of $\mathfrak{h}$ on the ring $\mathbb{C}[N_+]$ induced by the adjoint action
$$H \times N_+ \to N_+ \qquad h \cdot x = hxh^{-1}$$
- The exponential map $\mathfrak{n}_+ \to N_+$ is an isomorphism therefore N_+ (and hence U) is an affine space $\simeq \mathbb{A}^{|\Phi_+|}$ where Φ_+ is the set of positive roots. The exponential map commutes with the adjoint action of H, we will call a system of coordinates $(y_\alpha)_{\alpha\in\Phi_+}$ **homogeneous** if for any $h \in \mathfrak{h}$
$$h \cdot y_\alpha = -\alpha(h) y_\alpha$$
We will only consider homogeneous system of coordinates.

Proposition 5.1.1. *One can choose a system of homogenous coordinates y_α such that the action of $\mathfrak{n}_+$ on $\mathbb{C}[N_+] = \mathbb{C}[y_\alpha]$ satisfies*
$$e_\alpha \cdot y_\alpha = 1 \qquad e_\alpha \cdot y_\beta = 0 \quad \textit{unless } \alpha \le \beta$$
(Recall that $\alpha \le \beta$ if and only if $\beta - \alpha$ is a positive root). In particular we may write

$$e_\alpha = \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} P^\alpha_\beta(y) \frac{\partial}{\partial y_\beta}$$

for some $P^\alpha_\beta(y) \in \mathbb{C}[y_\alpha]$. The polynomials P^α_β have weight $-\beta + \alpha$, in particular they cannot contain the variable y_β.

Proof. Consider any homogeneous system of coordinates y_α. It is clear that in the weight decomposition of $\mathbb{C}[N_+]$ only negative weights appear

$$\mathbb{C}[N_+] = \bigoplus_{\lambda \in -\mathbb{Z}_+\Phi_+} \mathbb{C}[N_+]_\lambda \oplus \mathbb{C}1$$

Consider now $\alpha \in \Phi_+$ we want to prove that $e_\alpha y_\beta = 0$ unless $\beta \le \alpha$. Consider the action of $\mathfrak{h}$ on $e_\alpha y_\beta$, we have

$$h \cdot e_\alpha y_\beta = [h, e_\alpha] y_\beta + e_\alpha(-\beta(h) y_\beta) = (\alpha - \beta)(h) e_\alpha y_\beta$$

so it has weight $\alpha - \beta$, this lies in $-\mathbb{Z}_+\Phi_+ \cup \{0\}$ if and only if $\beta \geq \alpha$. In addition if $\alpha = \beta$ we find that $e_\alpha y_\alpha$ has weight 0 so it must be a multiple of 1. It is not 0 by the $\mathfrak{sl}_2$-case (see the next example) so the variables y_α may be re-scaled in order to get

$$e_\alpha \cdot y_\alpha = 1 \qquad e_\alpha \cdot y_\beta = 0 \quad \text{unless } \alpha \le \beta$$

To check that the polynomials P^α_β have weight $-\beta + \alpha$ we just notice that for any $h \in \mathfrak{h}$

$$\alpha(h)\left(\frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} P^\alpha_\beta(y) \frac{\partial}{\partial y_\beta}\right) = [h, e_\alpha] = \alpha(h) a_\beta + \sum_{\beta>\alpha} \big(h \cdot P^\alpha_\beta(y) + \beta(h) P^\alpha_\beta(y)\big) \frac{\partial}{\partial y_\beta}$$

□

We give as an example an explicit exposition of the $\mathfrak{sl}_2$ case.

Example 5.1.1 ($\mathfrak{sl}_2$)**.** In the case of $\mathfrak{sl}_2$ and the its group SL_2 we have

$$G/B_- \simeq \mathbb{P}^1 \quad \text{and} \quad U = \mathbb{A}^1 = \{[u, -1]\} \subset \mathbb{P}^1$$

we have the following

$$\begin{pmatrix} 1 & -\epsilon \\ 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} u \\ -1 \end{bmatrix} = \begin{bmatrix} u+\epsilon \\ -1 \end{bmatrix} \qquad e \mapsto \frac{\partial}{\partial u}$$
$$\begin{pmatrix} 1-\epsilon & 0 \\ 0 & 1+\epsilon \end{pmatrix} \cdot \begin{bmatrix} u \\ -1 \end{bmatrix} = \begin{bmatrix} u(1-\epsilon) \\ -1-\epsilon \end{bmatrix} = \begin{bmatrix} u(1-2\epsilon) \\ -1 \end{bmatrix} \qquad h \mapsto -2u\frac{\partial}{\partial u}$$
$$\begin{pmatrix} 1 & 0 \\ -\epsilon & 1 \end{pmatrix} \cdot \begin{bmatrix} u \\ -1 \end{bmatrix} = \begin{bmatrix} u \\ -1-u\epsilon \end{bmatrix} = \begin{bmatrix} u(1-u\epsilon) \\ -1 \end{bmatrix} \qquad f \mapsto -u^2\frac{\partial}{\partial u}$$

We introduce a very convenient notation which we will extensively use in what follows: we denote by $a^*_\alpha := y_\alpha$ and by $a_\alpha := \frac{\partial}{\partial y_\alpha}$.

By the considerations made so far, choosing an homogeneous coordinate system a^*_α such that $e_\alpha a^*_\alpha = 1$ we obtain the following formulas for the action of $\mathfrak{g}$ on N_+

$$\begin{aligned} e_i &\mapsto a_{\alpha_i} + \sum_{\beta > \alpha_i} P^i_\beta(a^*) a_\beta \\ h_i &\mapsto \sum_\beta -\beta(h_i) a^*_\beta a_\beta \\ f_i &\mapsto \sum_\beta Q^i_\beta(a^*) a_\beta \end{aligned}$$

The isomorphism $U \simeq N_+$ allows us to consider the action of right multiplication of N_+ on U. This induces a anti-homomorphism of Lie algebras

$$\mathfrak{n}_+ \to \mathrm{Der}(N_+)$$

we will write e^R_α for the operator on $\mathbb{C}[N_+]$ associated to $-e_\alpha$ with this right action, so that the map $e_\alpha \mapsto e^R_\alpha$ is a Lie algebra homomorphism. This action will be very useful to us and therefore we will keep considering alongside the left action of $\mathfrak{g}$.

Remark 5.1.2. We list a couple of useful remarks concerning this right action:

- The right action of $\mathfrak{n}_+$ commutes with the left action of $\mathfrak{n}_+$:

  $$[e_\alpha, e^R_\beta] = 0$$

  this follows from the fact that the action of left multiplication and the action of right multiplication on N_+ commute;

- The action of right multiplication on N_+ is part of a bigger action of B_+ constructed considering the isomorphism $N_+ \subset U^R \subset B_- \setminus G$, the induced right action of $\mathfrak{h}$ is exactly '$-$' the action we already knew. In particular

  $$[h, e^R_\alpha] = \alpha(h) e^R_\alpha$$

  and therefore the operators e^R_α have a description similar to the operators e_α:

  $$e^R_\alpha = \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} P^{R,\alpha}_\beta(y) \frac{\partial}{\partial y_\beta}$$

5.1.1 $\mathbb{C}[N_+]$ as a Verma module

We describe the $\mathfrak{g}$ module $\mathbb{C}[N_+]$ defined above as a coinduced module.

Definition 5.1.1. Let $\chi \in \mathfrak{h}^*$ be a character. Consider the action of $\mathfrak{b}_-$ on $\mathbb{C}_\chi$ where $\mathfrak{h}$ acts as χ while $\mathfrak{n}_-$ acts like 0. We define the **Verma module** as the induced module

$$M_\chi := \mathrm{Ind}\,^{\mathfrak{g}}_{\mathfrak{b}_-} \mathbb{C}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_-)} \mathbb{C}_\chi$$

it is isomorphic, both as a vector space and a $\mathfrak{n}_+$ module, to $U(\mathfrak{n}_+)$.

Note that the the algebra $U(\mathfrak{n}_+)$ has a decomposition induced by the action of $\mathfrak{h}$:

$$U(\mathfrak{n}_+) = \bigoplus_{\gamma \in Q_+} U(\mathfrak{n}_+)_\gamma$$

where $Q_+ = \mathbb{Z}_+ \Phi_+$ and $U(\mathfrak{n}_+)_\gamma = \{x \in U(\mathfrak{n}_+) : h \cdot x = \gamma(h) x \text{ for any } h \in \mathfrak{h}^*\}$.

We define the **contragradient Verma module** M^*_χ as

$$M^*_\chi := \mathrm{Coind}\,^{\mathfrak{g}}_{\mathfrak{b}_-} \mathbb{C}_\chi = \mathrm{Hom}^{res}_{U(\mathfrak{b}_-)}(U(\mathfrak{g}), \mathbb{C}_\chi)$$

where $\mathrm{Hom}^{res}_{U(\mathfrak{b}_-)}(U(\mathfrak{g}), \mathbb{C}_\chi)$ is the space of $U(\mathfrak{b}_-)$ linear maps $U(\mathfrak{g}) \to \mathbb{C}_\chi$ which are supported on finitely many factors of the direct sum

$$U(\mathfrak{g}) = \bigoplus_{\gamma \in Q_+} U(\mathfrak{b}_-) \otimes U(\mathfrak{n}_+)_\gamma$$

Verma modules are described easily, thanks to their definition. They satisfy a lot of useful properties. For instance for any $\mathfrak{g}$ module N we have

$$\mathrm{Hom}_{\mathfrak{b}_-}(\mathbb{C}_\chi, N) = \mathrm{Hom}_{\mathfrak{g}}(M_\chi, N) \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{b}_-}(N, \mathbb{C}_\chi) = \mathrm{Hom}_{\mathfrak{g}}(N, M_\chi^*)$$

Proposition 5.1.2. *The $\mathfrak{g}$ module $\mathbb{C}[N_+]$ is isomorphic to M_0^*.*

Proof. Consider the pairing

$$U(\mathfrak{n}_+) \times \mathbb{C}[N_+] \to \mathbb{C} \qquad (x, P) \mapsto (x \cdot P)(1)$$

This in $\mathfrak{n}_+$ invariant as well as $\mathfrak{h}$ invariant with respect to the vector field action on the second factor and to the action by minus right multiplication of $\mathfrak{n}_+$ and the adjoint action of $\mathfrak{h}$ on $U(\mathfrak{n}_+)$. We can prove by induction that given any polynomial $P \in \mathbb{C}[N_+]$ there exists an element $U \in U(\mathfrak{n}_+)$ such that $U \cdot P = 1$. Indeed consider the maximal root α_{max} appearing in P, then a sufficient iterate application of $e_{\alpha_{max}} \cdot P$ eliminates this variable and we may proceed eliminating all other variables. The pairing above is therefore non degenerate. In particular notice that under the decompositions

$$U(\mathfrak{n}_+) = \bigoplus_{\gamma \in Q_+} U(\mathfrak{n}_+)_\gamma \qquad \mathbb{C}[N_+] = \bigoplus_{\gamma \in Q_+} \mathbb{C}[N_+]_{-\gamma}$$

It is quite clear, by $\mathfrak{h}$ invariancy that under this pairing $\mathbb{C}[N_+]_{-\gamma} \to U(\mathfrak{n}_+)^*_\gamma$ in particular we obtain

$$\mathbb{C}[N_+] \to U(\mathfrak{n}_+)^\vee := \bigoplus_{\gamma \in Q_+} U(\mathfrak{n}_+)^*_\gamma$$

This is an isomorphism since it is injective and by equality of dimensions of the spaces $\mathbb{C}[N_+]_{-\gamma}$ and $U(\mathfrak{n}_+)_\gamma$. Now consider the morphism

$$\mathbb{C}[N_+] \to \mathbb{C}_0 \qquad P \mapsto P(1)$$

this is easily seen to be a morphism of $\mathfrak{b}_-$ modules since we are considering the action of $\mathfrak{g}$ on $N_+ = U \subset G/B_-$ and therefore for any $a \in \mathfrak{b}$ we have $a \cdot 1 = 1$. This morphism, since the coinduction functor is the right adjoint of the restriction functor and since $M_0^* = \mathrm{Coind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_0$, induces a morphism of $\mathfrak{g}$ modules $\mathbb{C}[N_+] \to M_0^*$ using the identification of $\mathfrak{n}_+$ modules $M_0^* = U(\mathfrak{n}_+)^\vee$ we find that this homomorphism coincides with the one induced by the pairing above and therefore it is an isomorphism. $\square$

As a corollary, using the identification of $\mathfrak{n}_+$ modules $\mathbb{C}[N_+] = U(\mathfrak{n}_+)^\vee$, we may define other structures of $\mathfrak{g}$ module on $\mathbb{C}[N_+]$ defining

$$\mathbb{C}[N_+]_\chi := M_\chi^* \qquad \text{for } \chi \in \mathfrak{h}^*$$

These other $\mathfrak{g}$ structures may be obtained in a different way. Consider the Weyl algebra of differential operators on $N_+ = U$:

$$D(U) = D(N_+) := \mathbb{C}\left[y_\alpha, \frac{\partial}{\partial y_\alpha}\right]$$

It may be described as the free algebra in the variables $a^*_\alpha = y_\alpha, a_\alpha = \frac{\partial}{\partial y_\alpha}$ subject to the relations

$$[a^*_\alpha, a^*_\beta] = 0 \qquad [a_\alpha, a_\beta] = 0 \qquad [a_\alpha, a^*_\beta] = \delta_{\alpha,\beta}$$

This algebra carries a natural filtration $D_{\leq n}$ defined by the order of the differential operators. $D_{\leq 1}$ is easily seen to be a Lie subalgebra of D, it is spanned by elements of the form

$$\sum_\alpha P_\alpha(y)\frac{\partial}{\partial y_\alpha} + Q(y)$$

In addition we have a short exact sequence of Lie algebras

$$0 \to \mathbb{C}[U] \to D_{\leq 1} \to \mathrm{Vect}(U) \to 0$$

The map $D_{\leq 1} \to \mathrm{Vect}(U)$ is not defined trough the natural action of D on the space of functions $\mathbb{C}[U] = \mathbb{C}[N_+]$ but as the action of $D_{\leq 1}$ induced by the bracket on its abelian ideal $\mathbb{C}[U]$.

Note that this sequence is actually split, since we may identify $\mathrm{Vect}(N_+)$ with the Lie subalgebra of differential operators of order ≤ 1 which kill the constant function $1 \in \mathbb{C}[N_+]$ with the natural action of $D_{\leq 1}$ on $\mathbb{C}[N_+]$.

Lemma 5.1.1. *Let*

$$0 \to L_1 \to L_2 \to L_3 \to 0$$

be a split exact sequence of Lie algebras, and suppose L_1 *is an abelian ideal of* L_2*. Note that the adjoint action of* L_2 *on* L_1 *induces an action of the quotient* L_3 *(since* L_1 *is abelian). Therefore* L_1 *is naturally an* L_3 *module.*

Let $\mathfrak{g}$ *be another Lie algebra and suppose we are given a morphism* $\mathfrak{g} \to L_3$*. This makes* L_1 *into a* $\mathfrak{g}$ *module. Then the set of liftings up to isomorphism*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & 0 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

is in bijection with

$$H^1(\mathfrak{g}, L_1)$$

Using the above lemma, and noticing that by the Shapiro lemma

$$H^1(\mathfrak{g}, \mathrm{Coind}\,^{\mathfrak{g}}_{\mathfrak{b}_-}\mathbb{C}_0) = H^1(\mathfrak{b}_-, \mathbb{C}_0) = (\mathfrak{b}_-/[\mathfrak{b}_-, \mathfrak{b}_-])^* = \mathfrak{h}^*$$

It is not difficult to see that composing the lifting $\mathfrak{g} \to D_{\leq 1}$ with the natural action of $D_{\leq 1}$ on $\mathbb{C}[N_+]$ we obtain the structure of $\mathfrak{g}$ module M^*_χ on $\mathbb{C}[N_+]$.

5.1.2 Explicit formulas

Let $(\alpha_i)_{i=1,\ldots,l}$ the chosen basis Δ for the root system Φ and let e_i, h_i, f_i be the induced standard set of generators for $\mathfrak{g}$. The action of $\mathfrak{g}$ on $\mathbb{C}[N_+] = M^*_\chi$ may be written, with respect to the algebra

$D_{\leq 1} = \mathbb{C}[a^*_\alpha, a_\alpha]$

$$e_i \mapsto a_{\alpha_i} + \sum_{\beta > \alpha_i} P^i_\beta(a^*) a_\beta \tag{5.1}$$

$$h_i \mapsto \sum_\beta -\beta(h_i) a^*_\beta a_\beta + \chi(h_i) \tag{5.2}$$

$$f_i \mapsto \sum_\beta Q^i_\beta(a^*) a_\beta + \chi(h_i) a^*_{\alpha_i} \tag{5.3}$$

This may be directly checked using the definition of M^*_χ and comparing it with $M^*_0 = \mathbb{C}[N_+]$.

5.2 The case of $\hat{\mathfrak{g}}_\kappa$

5.2.1 Overview

We now turn to our case of interest. We are going to define the analogous notions of algebra of differential operators and of vector fields in the loop setting and in the vertex algebra setting. Our first goal will be to find a 'loop version' of the exact sequence we encountered in the finite dimensional case, we will find a huge difference though: the sequence is non split and it is not possible to lift the morphism $\mathfrak{g}((t)) \to \mathrm{Vect}(LU)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}[LU] & \longrightarrow & A^{\mathfrak{g}}_{\leq 1} & \longrightarrow & \mathrm{Vect}(LU) & \longrightarrow & 0 \\ & & & & & \nwarrow \nexists & \uparrow & & \\ & & & & & & \mathfrak{g}((t)) & & \end{array}$$

This difficulty is partially resolved: there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}[LU] & \longrightarrow & A^{\mathfrak{g}}_{\leq 1} & \longrightarrow & \mathrm{Vect}(LU) & \longrightarrow & 0 \\ & & {\scriptstyle \mathbf{1}\mapsto 1}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C}\mathbf{1} & \longrightarrow & \hat{\mathfrak{g}}_{\kappa_c} & \longrightarrow & \mathfrak{g}((t)) & \longrightarrow & 0 \end{array}$$

the critical value κ_c enters the picture again.

5.2.2 Action of $\mathfrak{g}((t))$ by vector fields

Consider the open subset $U \subset G/B_-$ as before. Applying the loop functor we obtain an action

$$LG \times L(G/B_-) \to L(G/B_-)$$

This action induces an action of $\mathfrak{g}((t))$, the Lie algebra of LG on the ring of functions $\mathbb{C}[LU]$ obtained by vector fields. We wish to describe this action in terms of the finite dimensional action $\mathfrak{g} \to \mathbb{C}[U]$.

Recall that the ring of functions on LU is isomorphic to

$$\mathbb{C}[LU] = \varprojlim \frac{\mathbb{C}[a^*_{\alpha,n}]_{\alpha\in\Phi_+, n\in\mathbb{Z}}}{(a^*_{\alpha,n})_{n\geq N}}$$

We change a little bit our usual notation and define $a^*_{\alpha,n}$ to be the regular function

$$a^*_{\alpha,n} : LU(R) \to \mathbb{A}^1(R) \qquad (x_\beta(t))_\beta = (\sum_{n\in\mathbb{Z}} x_{\beta,n}t^{-n})_\beta \mapsto x_{\alpha,n}$$

In addition let $a_{\beta,m}$ be the vector field defined by

$$a_{\alpha,m} : LU(R) \to TLU(R) \qquad (r_\beta(t))_\beta = (\sum_n r_{\beta,n}t^{-n})_\beta \mapsto \big(\sum_n (r_{\beta,n} + \epsilon\delta_{\beta,\alpha}\delta_{n,-m})t^{-n}\big)_\beta$$

Note that following the definitions we have that the action of the vector fields $a_{\beta,m}$ on the functions $a^*_{\alpha,n}$ is given by the following:

$$a_{\beta,m} \cdot a^*_{\alpha,n} = \delta_{\alpha,\beta}\delta_{n,-m} \in \mathbb{C}[LU]$$

We proceed describing the Lie algebra of vector fields on LU. We denote by $a_{\alpha,n}$ the vector field

Proposition 5.2.1. $\mathrm{Vect}(LU)$ *is a Lie algebra isomorphic to the Lie algebra of formal series in the variables* $a^*_{\alpha,n}, a_{\beta,m}$*, of the form*

$$\sum_{\beta,n\in\mathbb{Z}} P_{\beta,n}(a^*)a_{\beta,n} \qquad P_{\beta,n}(a^*) \in \mathbb{C}[LU]$$

for which the $P_{\alpha,n}$ *satisfy the following property: for each* $N \geq 0$ *there exists a* $K \geq N$ *such that each* P_n *for* $n \leq -K$ *is in the ideal generated by* $(a^*_{\alpha,n})_{n\geq N}$*. The bracket is defined as*

$$\big[\sum_{\beta,n} P_{\beta,n}(a^*)a_{\beta,n}, \sum_{\gamma,m} Q_{\gamma,m}(a^*)a_{\gamma,m}\big] = \sum_{\beta,\gamma,n,m} \big(P_{\beta,n}\frac{\partial Q_{\gamma,m}}{\partial a^*_{\beta,-n}}a_{\gamma,m} - \frac{\partial P_{\beta,n}}{\partial a^*_{\gamma,-m}}Q_{\gamma,m}a_{\beta,n}\big)$$

Proof. Let $v : LU \to TLU$ be a vector field. Consider the coordinates $a^*_{\alpha,n}, a^{*,\epsilon}_{\alpha,n}$ on TLU defined on R points as

$$a^*_{\alpha,n}\big(\sum_{n\in\mathbb{Z}}(r_{\beta,n} + \epsilon r^\epsilon_{\beta,n})t^{-n}\big)_\beta \mapsto r_{\alpha,n}$$
$$a^{*,\epsilon}_{\alpha,n}\big(\sum_{n\in\mathbb{Z}}(r_{\beta,n} + \epsilon r^\epsilon_{\beta,n})t^{-n}\big)_\beta \mapsto r^\epsilon_{\alpha,n}$$

By hypothesis $a^*_{\alpha,n} \circ v = a^*_{\alpha,n}$ where the regular function on the right hand side is the usual coordinate on LU as defined above. Let $P_{\alpha,-n} := a^{*,\epsilon}_{\alpha,n} \circ v$, it is a regular function on LU. Note that by construction the vector field v is described as follows (on R points)

$$v(\sum_{n\in\mathbb{Z}} r_{\beta,n}t^{-n})_\beta = \big(\sum_{n\in\mathbb{Z}}(r_{\beta,n} + \epsilon P_{\beta,-n}(r))t^{-n}\big)_\beta$$

We write

$$v = \sum_{\beta,n\in\mathbb{Z}} P_{\beta,n}a_{\beta,n}$$

The condition for such a function v to be well defined on LU is that the series on the right hand side has to be a Laurent series. Consider now the subspace $L_N U$ which is affine with ring of coordinates

$$A = \mathbb{C}[x_{\alpha,n}]_{n<N}$$

And consider the its universal element

$$X_N \in L_N U(A)$$

this must be mapped through ν to a Laurent series, let's say of pole at most M. It follows from the Yoneda lemma that for any $\mathbb{C}$ algebra R and any element $x \in L_N U(R)$ the Laurent series $\nu(x)$ has pole at most M. From this discussion easily follows the condition on the $P_{\alpha,n}$ stated in the proposition.

On the other hand it may be directly checked that each formal series of the above form induces a well defined vector field. The computation of the bracket is straightforward. □

Consider now the formal power series

$$a^*_\alpha(z) := \sum_{n\in\mathbb{Z}} a^*_{\alpha,n} z^{-n} \qquad a_\alpha(z) = \sum_{n\in\mathbb{Z}} a_{\alpha,n} z^{-n-1}$$

Lemma 5.2.1. *The Lie algebra homomorphism $\mathfrak{g}((t)) \to \mathrm{Vect}(LU)$ is given by the following formula:*

$$e_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\beta>\alpha} P^i_\beta(a^*(z)) a_\beta(z) \tag{5.4}$$

$$h_i(z) \mapsto \sum_\beta -\beta(h_i) a^*_\beta(z) a_\beta(z) \tag{5.5}$$

$$f_i(z) \mapsto \sum_\beta Q^i_\beta(a^*(z)) a_\beta(z) \tag{5.6}$$

Proof. Consider the morphism induced by the action of G on G/B_-

$$\mathfrak{g} \times U \to TU$$

Let J_a be a basis for $\mathfrak{g}$ and let J^*_a be the basis for $\mathfrak{g}^*$ associated to the basis J_a. Suppose that J_a acts on $\mathbb{C}[U]$ by the vector field

$$\sum_\alpha P^a_\alpha(y) \frac{\partial}{\partial y_\alpha}$$

Let $\mathbb{C}[y_\alpha] = \mathbb{C}[U]$ and let $\mathbb{C}[y_\alpha, y^\epsilon_\alpha] = \mathbb{C}[TU]$. The expression above may be restated saying that induced morphism on the ring of functions

$$\mathbb{C}[y_\alpha, y^\epsilon_\alpha] \to \mathbb{C}[y_\alpha] \otimes \mathbb{C}[J^*_a]$$

is characterized by the formulas

$$y_\alpha \mapsto y_\alpha \qquad y^\epsilon_\alpha \mapsto \sum_a P^a_\alpha(y) \otimes J^*_a$$

This allows us to describe the vector field associated to an element $J_a \otimes t^n \in \mathfrak{g}((t))$.

For any $\mathbb{C}$ algebra R the map

$$\mathfrak{g}(R) \times U(R) \to TU(R)$$

is described thanks to the above remark by

$$(J^a \otimes r_0, (r_\alpha)_\alpha)) \mapsto (r_\alpha + \epsilon P^a_\alpha(r) r_0)_\alpha$$

applying this formula when $R = R((t))$ and $r_0 = t^n$ we find that the vector field associated to $J_a \otimes t^n$ is given, on R points by

$$(r_\alpha(t))_\alpha = \big(\sum_{n\in\mathbb{Z}} r_{\alpha,n}t^{-n}\big)_\alpha \mapsto \big(\sum_{n\in\mathbb{Z}} r_{\alpha,n}t^{-n} + \epsilon P^a_\alpha(r(t))t^n\big)_\alpha$$

If we write $P^a_\alpha(r(t))t^n = \sum_{m\in\mathbb{Z}} Q^a_{\alpha,m}(r)t^{m+n}$ then the $Q^a_{\alpha,m}$ actually define functions on LU and the associated vector field is

$$\sum_{\alpha,m} Q^a_{\alpha,m} a_{\alpha,m+n}$$

which is exactly the coefficient of z^{-n-1} of the expression $\sum_\alpha P^a_\alpha(a^*(z))a_\alpha(z)$. □

5.2.3 The completed Weyl algebra $\widetilde{A}^{\mathfrak{g}}$

Consider $A^{\mathfrak{g}}$ the algebra generated by elements $a^*_{\alpha,n}, a_{\beta,m}$ for $\alpha, \beta \in \Phi_+$ and $n, m \in \mathbb{Z}$ subject to the relations

$$[a^*_{\alpha,n}, a^*_{\beta,m}] = [a_{\alpha,n}, a_{\beta,m}] = 0 \qquad [a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{n,-m}$$

Consider a topology on $A^{\mathfrak{g}}$ generated by the subspaces $I_{N,M}$ the left ideals generated by $a_{\alpha,n}$: $n \geq N$ and $a^*_{\beta,m} m \geq M$.

Definition 5.2.1. Define the completed Weyl algebra $\widetilde{A}^{\mathfrak{g}}$ to be the completion of $A^{\mathfrak{g}}$ for the topology induced by the $I_{N,M}$. The product on $A^{\mathfrak{g}}$ may be checked to continuous for this topology and therefore induces a structure of associative algebra on $\widetilde{A}^{\mathfrak{g}}$.

An element of $\widetilde{A}^{\mathfrak{g}}$ may be written as an infinite sum

$$\sum_{n\geq N} P_{\alpha,N} a_{\alpha,n} + \sum_{m\geq M} Q_{\alpha,m} a^*_{\alpha,m} \qquad P_{\alpha,n}, Q_{\beta,m} \in A^{\mathfrak{g}}$$

Let $\widetilde{A}^{\mathfrak{g}}_0$ be the abelian complete subalgebra generated by the $a^*_{\alpha,n}$, following the definitions is quite clear that $\widetilde{A}^{\mathfrak{g}}_0 \simeq \mathbb{C}[LU]$. Finally we define $\widetilde{A}^{\mathfrak{g}}_{\leq 1}$ to be the completion of the subspace $A^{\mathfrak{g}}_{\leq 1}$ which is defined to be the span of products of elements of $A^{\mathfrak{g}}_0$ and elements $a_{\alpha,n}$.

Let's describe in more detail $\widetilde{A}^{\mathfrak{g}}_{\leq 1}$. Note first that the subspace $A^{\mathfrak{g}}_{\leq 1}$ is naturally a Lie algebra, with the bracket induced by the structure of associative algebra on $A^{\mathfrak{g}}$. Since the product in $A^{\mathfrak{g}}$ is continuous so is this bracket, $\widetilde{A}^{\mathfrak{g}}_{\leq 1}$ carries therefore a natural structure of Lie algebra.

In addition a general element of $\widetilde{A}^{\mathfrak{g}}$ may be written as an infinite sum

$$\sum_{n\geq N} P_{\alpha,n} a_{\alpha,n} + \sum_{m\geq M_1}\sum_{k\in K_m} Q^1_{\alpha,\beta,k,m} a_{\beta,k} a^*_{\alpha,m} + \sum_{m\geq M_2} Q^2_{\alpha,m} a^*_{\alpha,m}$$

where $K_m \subset \mathbb{Z}$ is a finite set and $P_{\alpha,n}, Q^1_{\alpha,\beta,k,m}, Q^2_{\alpha,m} \in A^{\mathfrak{g}}_0$.

We are ready to define the exact sequence we are interested in

Lemma 5.2.2. *There exists an exact sequence of Lie algebras*

$$0 \longrightarrow \mathbb{C}[LU] \longrightarrow \widetilde{A}^{\mathfrak{g}}_{\leq 1} \longrightarrow \mathrm{Vect}(LU) \longrightarrow 0$$

Proof. We claim that the linear map that sends an element of the form

$$\sum_{n\geq N} P_{\alpha,n} a_{\alpha,n} + \sum_{m\geq M_1} \sum_{k\in K_m} Q^1_{\alpha,\beta,k,m} a_{\beta,k} a^*_{\alpha,m} + \sum_{m\geq M_2} Q^2_{\alpha,m} a^*_{\alpha,m}$$

to the vector field

$$\sum_{n\geq N} P_{\alpha,n} a_{\alpha,n} + \sum_{m\geq M_1} \sum_{k\in K_m} a^*_{\alpha,m} Q^1_{\alpha,\beta,k,m} a_{\beta,k}$$

is an homomorphism of Lie algebras whose kernel is exactly $\widetilde{A}^{\mathfrak{g}}_0 = \mathbb{C}[LU]$. As a linear maps it is clearly well defined and is kernel is $\widetilde{A}^{\mathfrak{g}}_0$. It remains to show that it is a Lie algebra homomorphism. To see this it suffices to notice that the map is continuous and it is a Lie algebra homomorphism on the quotients. Since the bracket on both (complete) spaces is continuous this implies that the map $\widetilde{A}^{\mathfrak{g}}_{\leq 1} \to \mathrm{Vect}(LU)$ is a Lie algebra homomorphism as well. □

To proceed with our goal of lifting the homomorphism $\mathfrak{g}((t)) \to \mathrm{Vect}(LU)$ we are going to need the 'local' versions of these Lie algebras. The vertex algebra approach will be crucial.

5.3 Vertex algebra interpretation

We are now going to define a vertex algebra $M_{\mathfrak{g}}$ closely related to the completed Weyl algebra $\widetilde{A}^{\mathfrak{g}}$.

Definition 5.3.1. Consider the $A^{\mathfrak{g}}$ module $M_{\mathfrak{g}}$, generated by a vector $|0\rangle$ and such that

$$a^*_{\alpha,n}|0\rangle = 0 \ \forall n \geq 1 \qquad a_{\alpha,n}|0\rangle = 0 \ \forall n \geq 0$$

Equivalently, consider the abelian subalgebra $A^{\mathfrak{g}}_+$ generated by the elements $a^*_{\alpha,n}, a_{\beta,m}$ with $n \geq 1$ and $m \geq 0$ and consider its trivial one dimensional module $\mathbb{C}|0\rangle$, finally define

$$M_{\mathfrak{g}} = \mathrm{Ind}^{A^{\mathfrak{g}}}_{A^{\mathfrak{g}}_+} \mathbb{C}|0\rangle = A^{\mathfrak{g}} \otimes_{A^{\mathfrak{g}}_+} \mathbb{C}|0\rangle$$

Notice that since $A^{\mathfrak{g}} = A^{\mathfrak{g}}_- \otimes A^{\mathfrak{g}}_+$ as a right $A^{\mathfrak{g}}_+$ module where $A^{\mathfrak{g}}_-$ is the abelian subalgebra generated by $a^*_{\alpha,n}, a_{\beta,m}$ for $n < 1$ and $m < 0$. We have

$$M_{\mathfrak{g}} \simeq A^{\mathfrak{g}}_-$$

as vector spaces.

We define a structure of $\mathbb{Z}_+$ graded vertex algebra on $M_{\mathfrak{g}}$ as follows:

- ($\mathbb{Z}_+$ grading) We set $\deg a_{\alpha_1,n_1} \dots a_{\alpha_k,n_k} a^*_{\beta_1,m_1} \dots a^*_{\beta_l,l}|0\rangle := -\sum n_i - \sum m_j$;
- (Vacuum vector) We set $|0\rangle$ as the vacuum vector;
- (Translation operator) We define the translation operator T by the formulas

$$T|0\rangle = 0 \quad [T, a_{\alpha,n}] = -n a_{\alpha,n-1} \quad [T, a^*_{\alpha,n}] = (-n-1) a^*_{\alpha,n-1}$$

- (Vertex operators) We set

$$Y(a_{\alpha,-1}|0\rangle, z) := a_\alpha(z) \qquad Y(a^*_{\alpha,0}|0\rangle, z) := a^*_\alpha(z)$$

$$Y(a_{\alpha_1,n_1}\dots a_{\alpha_k,n_k}a^*_{\beta_1,m_1}\dots a^*_{\beta_l,l}|0\rangle, z) =$$
$$\prod \frac{1}{(-n_i-1)!}\prod\frac{1}{(-m_j)!} : \partial_z^{-n_1-1}a_{\alpha_1}(z)\dots\partial_z^{-n_k-1}a_{\alpha_k}(z)\partial_z^{-m_1}a^*(z)_{\beta_1}\dots\partial_z^{-m_l}a^*_{\beta_l}(z)$$

Thanks to the reconstruction theorem to prove that this defines a structure of a vertex algebra it is enough to show that the fields $a_\alpha(z)$ and $a^*_\alpha(z)$ are mutually local.

$$[a_\alpha(z), a^*_\beta(w)] = \sum_{n,m}[a_{\alpha,n}, a^*_{\alpha,m}]z^{-n-1}w^{-m} = \sum_{n,m}\delta_{\alpha,\beta}\delta_{n,-m}z^{-n-1}w^{-m} = \delta_{\alpha,\beta}\delta(z-w)$$

As in the case of $V_\kappa(\mathfrak{g})$ and the enveloping algebra $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ we may interpret vertex operators on $M_\mathfrak{g}$ as vertex operators with coefficients in $\widetilde{A}^\mathfrak{g}$.

Proposition 5.3.1. *The complete algebra* $\tilde{U}(M_\mathfrak{g})$ *is canonically isomorphic to* $\widetilde{A}^\mathfrak{g}$*. In addition, the map*

$$U(M_\mathfrak{g}) \to \tilde{U}(M_\mathfrak{g})$$

is injective and hence $U(M_\mathfrak{g})$ *is naturally a Lie subalgebra of* $\widetilde{A}^\mathfrak{g}$*.*

Proof. The proof of this statement is completely analogous to the proof of theorem 4.2.2. □

We denote by $\widetilde{A}^\mathfrak{g}_{0,loc} := \widetilde{A}^\mathfrak{g}_0 \cap U(M_\mathfrak{g})$, by $\widetilde{A}^\mathfrak{g}_{\leq 1,loc} := \widetilde{A}^\mathfrak{g}_{\leq 1}\cap U(M_\mathfrak{g})$ and by $T_{loc} := \mathrm{Im}(\widetilde{A}^\mathfrak{g}_{\leq 1,loc} \to \mathrm{Vect}(LU))$. We have an exact sequence

$$0 \to \widetilde{A}^\mathfrak{g}_{0,loc} \to \widetilde{A}^\mathfrak{g}_{\leq 1,loc} \to T_{loc} \to 0$$

It is not difficult to see that $\widetilde{A}^\mathfrak{g}_{\leq 1,loc}$ is the span of the Fourier coefficients of operators of the form

$$Y[P(a^*)a_{\beta,-1}|0\rangle, z] = :P(a^*(z))a_{\beta(z)}:$$

By the definition of the morphism $\widetilde{A}^\mathfrak{g}_{\leq 1} \to \mathrm{Vect}(LU)$ such series are sent to the series of vector fields

$$P(a^*(z))a_\beta(z)$$

It follows that the image of $\mathfrak{g}((t))$ in $\mathrm{Vect}(LU)$ is actually contained in T_{loc}, we have therefore a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{A}^\mathfrak{g}_{0,loc} & \longrightarrow & \widetilde{A}^\mathfrak{g}_{\leq 1,loc} & \longrightarrow & T_{loc} & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & \mathfrak{g}((t)) & & \end{array}$$

5.4 Cocycles and liftings

Consider an exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \widetilde{\mathfrak{l}} \to \mathfrak{l} \to 0$$

where $\mathfrak{h}$ is an abelian ideal of $\widetilde{\mathfrak{l}}$. Since $\mathfrak{h}$ is an abelian ideal the adjoint action of $\widetilde{\mathfrak{l}}$ on $\mathfrak{h}$ factors through an action of $\mathfrak{l}$.

The Lie algebra $\widetilde{\mathfrak{l}}$ is called an extension of $\mathfrak{l}$. Choosing a splitting $\iota : \mathfrak{l} \to \widetilde{\mathfrak{l}}$ we can associate to it a 2-cocycle with coefficients in $\mathfrak{h}$

$$\omega(X, Y) := i([X, Y]_{\mathfrak{l}}) - [i(X), i(Y)]_{\widetilde{\mathfrak{l}}} \qquad \omega \in H^2(\mathfrak{l}, \mathfrak{h})$$

where $\mathfrak{h}$ has the $\mathfrak{l}$ module structure defined by the adjoint action of $\widetilde{\mathfrak{l}}$.

The following lemma will be crucial for us.

Lemma 5.4.1. *Let $\mathfrak{h}, \widetilde{\mathfrak{l}}, \mathfrak{l}$ as above and let $\omega \in H^2(\mathfrak{l}, \mathfrak{h})$ be a cocycle defining the extension.*

Suppose we are given another Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$ module $\mathfrak{h}'$ and a 2-cocycle $\sigma \in H^2(\mathfrak{g}, \mathfrak{h}')$ which defines an extension $\widetilde{\mathfrak{g}}$.

Suppose additionally that we are given two Lie algebra homomorphisms $\beta : \mathfrak{h}' \to \mathfrak{h}$ and $\alpha : \mathfrak{g} \to \mathfrak{l}$. Then if the cocycles

$$\beta_* \sigma \text{ and } \alpha^* \omega \in H^2(\mathfrak{g}, \mathfrak{h})$$

are equal, there exists a lifting $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{l}}$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \widetilde{\mathfrak{l}} & \longrightarrow & \mathfrak{l} & \longrightarrow & 0 \\
 & & \beta\uparrow & & \vdots\uparrow & & \uparrow\alpha & & \\
0 & \longrightarrow & \mathfrak{h}' & \longrightarrow & \widetilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

Our goal is to apply the above lemma in our setting, in particular we are going to consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A}^{\mathfrak{g}}_{0,loc} & \longrightarrow & \widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} & \longrightarrow & T_{loc} & \longrightarrow & 0 \\
 & & \iota\uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{\mathfrak{g}}_{\kappa_c} & \longrightarrow & \mathfrak{g}((t)) & \longrightarrow & 0
\end{array}$$

and denote by ω the cocycle in $H^2(\mathfrak{g}((t)), \widetilde{A}^{\mathfrak{g}}_{0,loc})$ obtained by pullback of the cocycle which defines the extension of the upper row, and by σ the cocycle defining the extension $\hat{\mathfrak{g}}_{\kappa_c}$. In what follows we will prove that

$$\iota_* \sigma = \omega \in H^2(\mathfrak{g}((t)), \widetilde{A}^{\mathfrak{g}}_{0,loc})$$

We will use the following lemma which is proved in [Fre07] lemma 5.6.7, which we state in a slightly different fashion.

Lemma 5.4.2. *If the two cocycles $\iota_*\sigma$ and ω, viewed as bilinear maps*

$$\bigwedge^2 \mathfrak{g}((t)) \to \widetilde{A}^{\mathfrak{g}}_{0,loc}$$

coincides when restricted to $\mathfrak{h}((t))$ they are actually equal as elements of

$$\iota_*\sigma = \omega \in H^2(\mathfrak{g}((t)), \widetilde{A}^{\mathfrak{g}}_{0,loc})$$

Proof. See [Fre07, Lemma 5.6.7]. □

5.4.1 Computation of the cocycle, the Wick formula

We proved before that the Lie algebra T_{loc} is spanned by the Fourier coefficients of series of the form

$$P(a^*(z))a_\beta(z)$$

choose the splitting $i : T_{loc} \to \widetilde{A}^{\mathfrak{g}}_{\leq 1,loc}$ which sends the n-th coefficient of the series above to the n-th coefficient of the normally ordered product

$$: P(a^*(z)) \cdot a_\beta(z) :$$

this is just a linear map which composed with $\widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \to T_{loc}$ is the identity.

To compute the cocycle must evaluate the expressions

$$[i(X), i(Y)] - i([X, Y]) \qquad \text{with } X, Y \in \mathfrak{g}((t))$$

In order to do this we state the so called Wick formula, which applies in our case, but is valid not only for the vertex algebra $M_{\mathfrak{g}}$ but for all the so called free field algebras.

Definition 5.4.1. A vertex algebra V is called a **free field algebra** if it is generated (in the sense of the reconstruction theorem) by fields a_α such that the coefficients $c_j(w)$ in the expansion

$$[a_\alpha(z), a_\beta(w)] = \sum_{m\geq 0} \frac{1}{m!} c_m(w) \partial_w^m \delta(z-w)$$

are constant (i.e. $c_m(w) \in \mathbb{C}[[w^{\pm 1}]] \subset \operatorname{End} V[[w^{\pm 1}]]$).

For a moment we treat the case $\mathfrak{g} = \mathfrak{sl}_2$, so that we may eliminate the index α, the general case is treated no differently. Therefore in what follows we write a^* instead of a^*_α and a instead of a_α.

Consider the OPEs

$$a(z)a^*(w) = \frac{1}{z-w} + :a(z)a^*(w): \tag{5.7}$$

$$a^*(z)a(w) = -\frac{1}{z-w} + :a^*(z)a(w): \tag{5.8}$$

$$\partial_z^n a^*(z)\partial_w^m a(w) = (-1)^n \frac{(n+m)!}{(z-w)^{n+m+1}} + :\partial_z^n a^*(z)\partial_w^m a(w): \tag{5.9}$$

$$\partial_z^m a(z)\partial_w^n a^*(w) = (-1)^{m+1} \frac{(n+m)!}{(z-w)^{n+m+1}} + :\partial_z^m a(z)\partial_w^n a^*(w): \tag{5.10}$$

Where as usual we consider the expansion of $\frac{1}{z-w}$ in positive powers of w.

Our goal is to give a formula to compute the product $P(z)Q(w)$ where P and Q are two **normally ordered monomials** in the formal series $\partial_z n^a(z), \partial_z^m a^*(z)$ (resp. w). This formula will express this product in terms of simpler normally ordered product.

A **single pairing** between $P(z)$ and $Q(w)$ is the choice of an ordered couple of the form $(\partial_z^n a^*(z), \partial_z^m a(w))$ (or $(\partial_z^n a(z), \partial_z^m a^*(w))$ where $\partial_z^n a^*(z)$ appears as a factor of $P(z)$ and $\partial_w^m a(w)$ appears as a factor of $Q(w)$. We attach to such a pairing the rational function

$$f_{n,m}(z,w) = (-1)^n \frac{(n+m)!}{(z-w)^{n+m+1}}$$

which appears in the OPE of the two factors. Note that a single pairing may appear multiple times.

A **multiple pairing** is a disjoint union of single pairings. To a multiple pairing B we associate the rational function $f_B(z,w)$ which is the product of all the rational functions of the single pairings appearing in B.

We define $(P(z)P(w))_B$ as the product of the two polynomials after we remove all the factors contained in the pairing, if all the factors are contained in B we set $(P(z)Q(w))_B = 1$. Finally we define the **contraction** of P and Q for the pairing B as the normally ordered product $: (P(z)Q(w))_B :$ multiplied by the function $f_B(z,w)$, in addition we define also the contraction for the empty pairing as

$$: P(z)Q(w) :_{\emptyset}: \quad := \quad : P(z)Q(w) :$$

Lemma 5.4.3 (Wick formula)**.** *The following equality holds*

$$P(z)Q(w) = \sum_{B \in \textit{pairings}} : P(z)Q(w) :_B$$

where we sum over all pairings B, including the empty one, counted with multiplicity. This statement holds also if we consider P and Q as series with coefficients in $\widetilde{A}^{\mathfrak{g}}$.

Proof. See [Kac98, Theorem 3.3]. □

We are going to use the Wick formula in the particular case of polynomials of the form $: P(z)a(z) :$ and $: Q(w)a(w) :$ where P and Q are monomials in the $\partial_z^n a^*(z)$ only. Monomials of this type are exactly the vertex operators of monomials of the form Pa_{-1} and Qa_{-1} respectively, with P and Q monomials in the variables a_n^* $n \leq 0$.

This of course moves us a step further in the calculation of the cocycle since the product $Y(Pa_{-1},z)Y(Qa_{-1},w)$ contains all the information of the commutator $[Y(Pa_{-1},z), Y(Qa_{-1},w)]$.

Lemma 5.4.4. *The following formula holds*

$$\begin{aligned}
Y(Pa_{-1},z)Y(Qa_{-1},w) = &: Y(Pa_{-1},z)Y(Qa_{-1},w) : \\
&+ \sum_{n \geq 0} \frac{1}{(z-w)^{n+1}} : Y(P,z)Y\left(\frac{\partial Q}{\partial a^*_{-n}} a_{-1}, w\right) : \\
&- \sum_{n \geq 0} \frac{1}{(z-w)^{n+1}} : Y\left(\frac{\partial P}{\partial a^*_{-n}} a_{-1}, z\right)Y(Q,w) : \\
&- \sum_{n,m \geq 0} \frac{1}{(z-w)^{n+m+2}} : Y\left(\frac{\partial P}{\partial a^*_{-n}}, z\right)Y\left(\frac{\partial Q}{\partial a^*_{-m}}, w\right) :
\end{aligned}$$

Proof. It suffices to apply the Wick formula, after noticing that the contraction with respect to the pair $(\partial_z^n a^*(z), a(w))$ (or $(a(z), \partial_w^m a^*(w))$), counted with multiplicity, corresponds to taking the derivative of P with respect to a^*_{-n} and to eliminate the factor a_{-1} from Qa_{-1}. □

We are ready to state a compact formula for our two cocycle.

Proposition 5.4.1. *The cocycle* ω *for the extension*

$$0 \longrightarrow \widetilde{A}^{\mathfrak{g}}_{0,loc} \longrightarrow \widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \longrightarrow T_{loc} \longrightarrow 0$$

calculated through the splitting i *defined by the normal ordering is given by the following formula*

$$\omega((Pa_{-1})_{[k]}, (Qa_{-1})_{[s]}) = \\ - \sum_{n,m\geq 0} \int \left(\frac{1}{(n+m+1)!} \partial_z^{n+m+1} Y(\frac{\partial P}{\partial a^*_{-n}}, z) Y(\frac{\partial Q}{\partial a^*_{-m}}, w) z^k w^s \right)_{|z=w} dw$$

Proof. We compute first the commutator between the vector fields formal series $P(z)a(z)$ and $Q(w)a(w)$. We claim that it is equal to

$$[P(z)a(z), Q(w)a(w)] = \sum_{n\geq 0} \partial_w^n \delta(z-w) P(z) \frac{\partial Q}{\partial a^*_{-n}}(w) a(w) - \sum_{n\geq 0} \partial_z^n \delta(z-w) \frac{\partial P}{\partial a^*_{-n}}(z) Q(w) a(z)$$

which is just a verification using the fact that

$$[a(z) \partial_w^n a^*(w)] = \partial_w^n \delta(z-w) \qquad [a(w), \partial_z^n a^*(z)] = \partial_z^n \delta(z-w)$$

In addition the Wick formula implies that the commutator between the series $Y(Pa_{-1}, z)$ and $Y(Qa_{-1}, w)$ is given by

$$\begin{aligned}[Y(Pa_{-1}, z) Y(Qa_{-1}, w)] = & \sum_{n\geq 0} \partial_w^n \delta(z-w) : Y(P, z) Y\left(\frac{\partial Q}{\partial a^*_{-n}} a_{-1}, w \right) : \\ & - \sum_{n\geq 0} \partial_w^n \delta(z-w) : Y\left(\frac{\partial P}{\partial a^*_{-n}} a_{-1}, z \right) Y(Q, w) : \\ & - \sum_{n,m\geq 0} \partial_w^{n+m+1} \delta(z-w) : Y\left(\frac{\partial P}{\partial a^*_{-n}}, z \right) Y\left(\frac{\partial Q}{\partial a^*_{-m}}, w \right) :\end{aligned}$$

It is quite clear that the first two factors of this expression are exactly the normally ordered product of the commutator $[P(z)a(z), Q(w)a(w)]$. What remains is the last row, whose (k, l) coefficient is given exactly by the formula we wanted. □

Corollary 5.4.1. *The extension*

$$0 \longrightarrow \widetilde{A}^{\mathfrak{g}}_{0,loc} \longrightarrow \widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \longrightarrow T_{loc} \longrightarrow 0$$

is non split.

Proof. We compute the cocycle for the elements $\mathbf{h}_n$ defined by the formula

$$Y(a_0^* a_{-1}, z) = \; :a^*(z)a(z): \; = \sum_{n\in\mathbb{Z}} \mathbf{h}_n z^{-n-1}$$

so that

$$\mathbf{h}_n = \sum_{k\in\mathbb{Z}} :a_k^* a_{n-k}:$$

According to the formula of proposition 5.4.1 we have

$$\omega(\mathbf{h}_n, \mathbf{h}_m) = -\int (\partial_z z^n w^m)_{|z=w} dw = -n\delta_{n,-m}$$

On the other hand we find that the image of $\mathbf{h}_n$ in T_{loc} is the vector field $\overline{\mathbf{h}}_n : \sum_{n\in\mathbb{Z}} a_k^* a_{n-k}$. It is quite easy to see that these vector fields commute with each other.

If the extension was split there would exist correction terms $f_n \in \widetilde{A}^{\mathfrak{g}}_{0,loc}$ such that

$$[\mathbf{h}_n + f_n, \mathbf{h}_m + f_m] = n\delta_{n,-m} + \overline{\mathbf{h}}_n \cdot f_m - \overline{\mathbf{h}}_m \cdot f_n = 0$$

But the $\overline{h}_n$ are 'linear' vector fields (i.e. linear in the coordinates a^*_{-k} on LU), therefore they cannot produce such a constant function. □

The elements $\mathbf{h}_n$ are not arbitrarily chosen: they are exactly the normally ordered vector fields corresponding to the action of $\mathfrak{h}((t))$,

Proposition 5.4.2. *Let $\omega \in H^2(\mathfrak{g}((t)), \widetilde{A}^{\mathfrak{g}}_{0,loc})$ be the cocycle obtained by pullback of the cocycle induced by the splitting i for the upper row, and let σ the cocycle defining the central extension $\hat{\mathfrak{g}}_{\kappa_c}$. Then ω and $\iota_*\sigma$ are equal if restricted to $\mathfrak{h}((t))$.*

Proof. The formula of proposition 5.4.1, applied to the vector fields

$$h_i(z) = -\sum_{\beta} \beta(h_i) a^*_\beta(z) a_\beta(z)$$

gives us by bilinearity, as in corollary 5.4.1

$$\omega(h_{i,n}, h_{j,n}) = \sum_{\beta,\gamma} \beta(h_i)\gamma(h_j)\delta_{\gamma,\delta}(-n\delta_{n,-m}) = -n\delta_{n-m} \sum_{\beta\in\Phi_+} \beta(h_i)\beta(h_j)$$

Since h_i acts on the subspace $\mathfrak{g}_\beta$ by multiplication of $\beta(h_i)$ it is clear that the expression above equals to

$$-n\delta_{n,-m}\frac{1}{2}\kappa_{\mathfrak{g}}(h_i, h_j) = \iota_*\sigma(h_i, h_j)$$

□

Combining this result with lemma 5.4.2 and with lemma 5.4.1 we obtain the following proposition.

Proposition 5.4.3. *There exists an homomorphism of Lie algebras* $\hat{\mathfrak{g}}_{\kappa_c} \to \widetilde{A}^{\mathfrak{g}}_{\leq 1, \mathrm{loc}}$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{A}^{\mathfrak{g}}_{0,\mathrm{loc}} & \longrightarrow & \widetilde{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}} & \longrightarrow & T_{\mathrm{loc}} & \longrightarrow & 0 \\ & & \iota\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{\mathfrak{g}}_{\kappa_c} & \longrightarrow & \mathfrak{g}((t)) & \longrightarrow & 0 \end{array}$$

5.4.2 Explicit formulas

Theorem 5.4.1. *There exists constants* $c_i \in \mathbb{C}$ *such that the homomorphism* $\hat{\mathfrak{g}}_{\kappa_c} \to \mathfrak{g}((t))$ *is given by the following formulas*

$$e_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\beta > \alpha_i} : P^i_\beta(a^*(z)) a_\beta(z) :$$

$$h_i(z) \mapsto - \sum_{\beta \in \Phi_+} \beta(h_i) : a^*_\beta(z) a_\beta(z) :$$

$$f_i(z) \mapsto \sum_{\beta \in \Phi_+} : Q^i_\beta(a^*(z)) a_\beta(z) : + c_i \partial_z a^*_{\alpha_i}(z)$$

Proof. See [Fre07, Theorem 6.1.3] □

In addition we may do all the previous steps considering the right action of $\mathfrak{n}_+$ as vector fields, the following proposition is quite clear from the construction.

Proposition 5.4.4. *The same construction performed for the right action* e^R_α *on* $\mathbb{C}[N_+]$ *a Lie algebra homomorphism*

$$L\mathfrak{n}_+ \to \widetilde{A}^{\mathfrak{g}}_{\leq 1} \qquad e^R_\alpha(z) \mapsto a_\alpha(z) + \sum_{\beta > \alpha_i} : P^{R,\alpha}_\beta(a^*(z)) a_\beta(z) :$$

which commutes with the fields $e_\alpha(z)$*.*

Note that by the initial remarks $P^{R,\alpha}_\beta$ does not contain the variable y_β and therefore there is no necessity of considering the normally ordered product.

5.5 Free field realization

Recall that in the finite dimensional case we could attach to any $\chi \in \mathfrak{h}^*$ a structure of $\mathfrak{g}$-module on $\mathbb{C}[U]$, obtained by modifying the standard action by vector fields and for which $\mathbb{C}[U] \simeq M^*_\chi$.

We are now going to construct an homomorphism of Lie algebras which 'glues togheter' all these different actions and then consider its vertex algebra analogue.

Consider the quotient G/N_- and the map

$$\pi : G/N_- \to G/B_-$$

this is a G equivariant morphism when we consider the action of G on both spaces by left multiplication. Let $\pi^{-1}(U)$ be the affine open subset of G/N_- defined by the preimage of $U \subset G/B_-$. The multiplication

$$N_+ \times H \to \pi^{-1}(U) \qquad (x, h) \mapsto xhN_-$$

induce an isomorphism. The morphism π, under the isomorphism $\pi^{-1}(U) = N_+ \times H$ and $U = N_+$ corresponds to the projection on the first factor.

As before the action of G induces an action of the Lie algebra $\mathfrak{g}$ by vector fields on the space $\pi^{-1}(U)$ therefore we obtain an homomorphism

$$\mathfrak{g} \to \mathrm{Vect}(U \times H) = \mathrm{Vect}(U) \otimes \mathbb{C}[H] \oplus \mathbb{C}[U] \otimes \mathrm{Vect}(H)$$

Denote by $b_i \in \mathrm{Vect}(H)$ the vector fields associated to $h_i \in \mathfrak{h}$ through the natural Lie algebra homomorphism

$$\mathfrak{h} \to \mathrm{Vect}(H)$$

We may describe the homomorphism $\mathfrak{g} \to \mathrm{Vect}(N_+ \times H)$ in terms of the homomorphism of the previous section $\mathfrak{g} \to \mathrm{Vect}(N_+)$.

Proposition 5.5.1. *The homomorphism* $\mathfrak{g} \to \mathrm{Vect}(U \times H)$ *may be described in terms of the coordinates* a^*_α *on* $U = N_+$ *and the vector fields* a_α *and* b_i *as follows*

$$\begin{aligned} e_i &\mapsto a_{\alpha_i} + \sum_{\beta > \alpha_i} P^i_\beta(a^*) a_\beta \\ h_i &\mapsto \sum_\beta -\beta(h_i) a^*_\beta a_\beta + b_i \\ f_i &\mapsto \sum_\beta Q^i_\beta(a^*) a_\beta + a^*_{\alpha_i} b_i \end{aligned}$$

Proof. We consider the isomorphism $U \times H = N_+ \times H$.

(e_i) It is clear that the action of left multiplication restricted to the subgroup N_+ coincides with the left multiplication on the first factor, hence the action of e_i is described by the same formulas we obtained when we studied the action by vector fields on $U = N_+$;

(h_i) Note that the action by left multiplication of G restricted to H sends $U \times H$ to itself. Since H normalizes N_+ we have that this action may be described as

$$h_1 \cdot (x, h_2) = (h_1 x h_1^{-1}, h_1 h_2)$$

It follows that the vector field associated to h_i is equal to the sum of the vector field induced by its action on N_+ plus the vector field induced by left multiplication on H which is exactly what we wanted.

(f_i) We identify $N_+ \times H$ with BB_-/N_-. We do some calculations considering a general element $g \in \mathfrak{n}_-(R) \subset N_-(R[\epsilon])$ let $(x, h) = xh \in N_+ \times H$ and write

$$g(xh) = (g{\cdot}x)\gamma(g,x)\beta(g,x)h \qquad \text{with } (g{\cdot}x) \in N_+(R[\epsilon]) \quad \text{and } \gamma(g,x) \in H(R[\epsilon]), \beta(g,x) \in N_-(R[\epsilon])$$

This shows that for any $f_\beta \in \mathfrak{n}_-$ the vector field associated to f_β is of the form

$$\sum_{\alpha\in\Phi_+} P^\beta_\alpha(a^*)a_\alpha + \sum_{i=1}^{l} Q^\beta_i(a^*)b_i$$

where the polynomials $Q^\beta_i(a^*)$ are functions on N_+. Notice that in particular for the generators f_i we must have that $Q^{\alpha_i}_j$ has weight $-\alpha_i$. Since the weight spaces for the simple roots are one dimensional it follows that there exist complex numbers $c_{ij} \in \mathbb{C}$ such that

$$Q^{\alpha_i}_j(a^*) = c_{ij}a^*_{\alpha_i}$$

By computing the bracket between the vector fields associated to e_i and f_j one easily finds that $c_{ij} = 0$ if $i \neq j$. Finally to compute c_{ii} we reduce ourselves to the $\mathfrak{sl}_2$ which we treat in the following proposition. □

Proposition 5.5.2 ($\mathfrak{sl}_2$ case)**.** *The homomorphism* $\mathfrak{sl}_2 \to \mathrm{Vect}(U \times H)$ *is described by the following formulas*

$$\begin{aligned} e &\mapsto a \\ h &\mapsto -2a^*a + b \\ f &\mapsto -(a^*)^2a + a^*b \end{aligned}$$

Proof. We just need to compute the action of f. We treat elements of $N \times H$ as classes in BB_-/B_-. Notice that the vector field b in the coordinate $\mu : H \to \mathbb{C}$ for which

$$h = \begin{pmatrix} \mu(h) & 0 \\ 0 & \mu(h)^{-1} \end{pmatrix}$$

we have $b = -\mu\partial_\mu$ now consider any $\lambda \in R, \mu \in R^*$. To compute the action of f as a vector field we need to computer

$$\begin{pmatrix} 1 & 0 \\ -\epsilon & 1 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & -\lambda(1-\epsilon\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu(1-\lambda\epsilon) & 0 \\ 0 & \mu^{-1}(1+\lambda\epsilon) \end{pmatrix} \right]$$

It is then clear that $\mu((-f)\cdot x) = \mu(x) - \epsilon\lambda(x)\mu(x)$ and therefore the b component of f is exactly λb. □

So the homomorphism $\mathfrak{g} \to \mathrm{Vect}(U \times H)$ has image contained in the subalgebra $\mathrm{Vect}(U) \oplus \mathbb{C}[U]\otimes\mathfrak{h}$. Considering the loop case, as in lemma 5.2.1 we obtain an homomorphism of Lie algebras

$$\mathfrak{g}((t)) \to \mathrm{Vect}(LU) \oplus \mathbb{C}[LU]\hat{\otimes}L\mathfrak{h}$$

where $\hat{\otimes}$ denotes the completed tensor product of the complete vector spaces $\mathbb{C}[LU], L\mathfrak{h}$. This homomorphism is described by the following formulas on the generators.

$$e_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\beta > \alpha_i} P^i_\beta(a^*(z)) a_\beta(z)$$

$$h_i(z) \mapsto \sum_\beta -\beta(h_i) a^*_\beta(z) a_\beta(z) + b_i(z)$$

$$f_i(z) \mapsto \sum_\beta Q^i_\beta(a^*(z)) a_\beta(z) + a^*_{\alpha_i}(z) b_i(z)$$

where $b_i(z) = \sum_{n\in\mathbb{Z}} b_{i,n} z^{-n-1}$ and $b_{i,n}$ is the vector field on LH induced by the action of $h_{i,n}$.

We want to emulate the work we did in the previous section in this new case, hence we would like to give a vertex algebra interpretation of these new fields. We define a new vertex algebra in full generality (even if for now we only need the simplest version of this algebra), the construction is analogous to the construction of $V_k(\hat{\mathfrak{g}})$

Definition 5.5.1. Let $\mathfrak{h}$ be the abelian finite dimensional Lie algebra with a basis $(b_i)_{i=1,\dots,l}$. Let κ be a bilinear anti-symmetric form on $\mathfrak{h}$. Consider $\hat{\mathfrak{h}}_\kappa$ the affine algebra associated to $\mathfrak{h}$ and κ. We denote its vacuum module

$$\pi_0^\kappa := V_\kappa(\mathfrak{h})$$

The following ordered monomials form a basis for π_0^κ

$$b_{i_1,n_1} \dots b_{i_m,n_m}|0\rangle \qquad n_j < 0$$

Note that for $\kappa = 0$ this vertex algebra is actually abelian, we denote it simply by π_0. This is the vertex algebra we are interested in for now.

To proceed in our quest to obtain a $\hat{\mathfrak{g}}_{\kappa_c}$ module we need to find an appropriate extension of the Lie algebra $\mathrm{Vect}(LU) \oplus \mathbb{C}[LU]\hat{\otimes}\mathrm{Vect}(LH)$. Note that this is actually a direct sum of Lie algebras (i.e. the first and the second factor commute).

We start by defining the local Lie algebras as in the previous case. Let $T_{loc} = \mathrm{Im}(\widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \to \mathrm{Vect}(LU))$ as before and let $I^{\mathfrak{g}}_{loc}$ to be the span of the Fourier coefficients of series of vector fields of the form

$$P(a^*(z)) b_i(z)$$

it is of course an abelian subalgebra of the Lie algebra $\mathbb{C}[LU]\hat{\otimes}\mathrm{Vect}(LH)$.

We consider the exact sequence

$$0 \longrightarrow \widetilde{A}^{\mathfrak{g}}_{0,loc} \longrightarrow \widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \oplus I^{\mathfrak{g}}_{loc} \longrightarrow T_{loc} \oplus I^{\mathfrak{g}}_{loc} \longrightarrow 0$$

and notice that the image of the homomorphism $\mathfrak{g}((t)) \to \mathrm{Vect}(LU) \oplus \mathbb{C}[LU]\hat{\otimes}\mathrm{Vect}(LH)$ is contained in the subalgebra $T_{loc} \oplus I^{\mathfrak{g}}_{loc}$.

Lemma 5.5.1. *The Lie algebra* $\widetilde{A}^{\mathfrak{g}}_{\leq 1,loc} \oplus I^{\mathfrak{g}}_{loc}$ *is naturally a Lie subalgebra of* $U(M_{\mathfrak{g}} \otimes \pi_0)$. *In addition* $M_{\mathfrak{g}} \otimes \pi_0$ *is still a free field algebra, therefore the Wick formula holds.*

Proof. This is completely analogous to the $M_{\mathfrak{g}}$ case. □

By the above lemma we may consider the splitting $i : T_{loc} \oplus I^{\mathfrak{g}}_{loc} \to \widetilde{A}^{\mathfrak{g}}_{\leq 1, loc} \oplus I^{\mathfrak{g}}_{loc}$ induced by taking the normally ordered product and compute the associated cocycle with the Wick formula.

Proposition 5.5.3. *The cocycle ω_{new} calculated with the above splitting is equal to the cocycle $\omega \in \mathrm{Hom}(\bigwedge^2 \mathfrak{g}((t)), \widetilde{A}^{\mathfrak{g}}_{0,loc})$ calculated in the previous section.*

In particular its restriction to $\mathfrak{h}((t))$ is equal to $\iota_* \sigma$ therefore there exists an homomorphism of Lie algebras

$$\hat{\mathfrak{g}}_{\kappa_c} \to \widetilde{A}^{\mathfrak{g}}_{\leq 1, loc} \oplus I^{\mathfrak{g}}_{loc} \subset U(M_{\mathfrak{g}} \otimes \pi_0)$$

5.5.1 Explicit formulas

The above morphism we obtained is described on the generators by the following formulas:

$$\begin{aligned}
e_i(z) &\mapsto a_{\alpha_i}(z) + \sum_{\beta > \alpha_i} : P^i_\beta(a^*(z)) a_\beta(z) : \\
h_i(z) &\mapsto - \sum_{\beta \in \Phi_+} \beta(h_i) : a^*_\beta(z) a_\beta(z) : + b_i(z) \\
f_i(z) &\mapsto \sum_{\beta \in \Phi_+} : Q^i_\beta(a^*(z)) a_\beta(z) : + c_i \partial_z a^*_{\alpha_i}(z) + a^*_{\alpha_i}(z) b_i(z)
\end{aligned}$$

5.5.2 Vertex algebra interpretation

Our next immediate goal is to translate the above homomorphism of Lie algebras in the vertex algebra language. The reason behind this and the vertex algebras we are considering are way smaller than the completed algebras we may consider instead (for instance the homomorphism $\hat{\mathfrak{g}}_{\kappa_c} \to U(M_{\mathfrak{g}} \otimes \pi_0)$ induces an homomorphism $\tilde{U}_{\kappa_c}(\hat{\mathfrak{g}}) \to \tilde{U}(M_{\mathfrak{g}} \otimes \pi_0)$) and therefore more manageable.

Lemma 5.5.2. *Let V be a $\mathbb{Z}$-graded vertex algebra. Then defining an homomorphism of $\mathbb{Z}$ graded vertex algebras*

$$V_k(\mathfrak{g}) \to V$$

is equivalent to the choice of vectors $\widetilde{J^a_{-1}}|0\rangle \in V$ such that the coefficients

$$Y_V(\widetilde{J^a_{-1}}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \widetilde{J^a_n} z^{-n-1}$$

satisfy the commutation relations of $\hat{\mathfrak{g}}_k$ with $\mathbf{1} = \mathrm{id}$.

Proof. Given an homomorphism of vertex algebras $\varphi : V_k(\mathfrak{g}) \to V$ we have

$$\begin{aligned}
[\widetilde{J^a_n}, \widetilde{J^b_m}] &= \sum_{k \geq 0} \binom{n}{k} \left(\widetilde{J^a_k} (\widetilde{J^b_{-1}} |0\rangle) \right)_{n+m-k} = \sum_{k \geq 0} \binom{n}{k} \varphi(J^a_k J^b_{-1} |0\rangle)_{n+m-k} \\
&= \binom{n}{0} \varphi([J^a, J^b]_{-1} |0\rangle)_{n+m} + \binom{n}{1} \varphi(\kappa(J^a, J^b)|0\rangle)_{n+m-1} \\
&= \widetilde{[J^a, J^b]}_{n+m} + n \kappa(J^a, J^b) \delta_{n,-m} \mathrm{id}_V
\end{aligned}$$

So the elements $\widetilde{J^a}_n$ satisfy the commutation relations of $\hat{\mathfrak{g}}_\kappa$.

On the other hand if we are given elements $\widetilde{J^a_{-1}}|0\rangle$ with the property above it is easily checked that the linear map $V_\kappa(\mathfrak{g}) \to V$ defined by

$$J^{a_1}_{n_1} \dots J^{a_m}_{n_m}|0\rangle \mapsto \widetilde{J^{a_1}_{n_1}} \dots \widetilde{J^{a_m}_{n_m}}|0\rangle$$

is an homomorphism of $\mathbb{Z}$ graded vertex algebras. □

Note that the operators of the explicit formulas, defining the action of $\hat{\mathfrak{g}}_k$ on $M_\mathfrak{g} \otimes \pi_0$ come actually from the vertex operators of certain elements in the vertex algebra $M_\mathfrak{g} \otimes \pi_0$.

$$Y\Big((a_{\alpha_i,-1} + P^i_\beta(a^*_0)a_{\beta,-1})|0\rangle, z\Big) = a_{\alpha_i}(z) + \sum_{\beta>\alpha_i} :P^i_\beta(a^*(z))a_\beta(z):$$

$$Y\Big(\big(-\sum \beta(h_i)a^*_{\beta,0}a_{\beta,-1} + b_{i,-1}\big)|0\rangle, z\Big) = -\sum_{\beta\in\Phi_+} \beta(h_i):a^*_\beta(z)a_\beta(z): + b_i(z)$$

$$Y\Big(\big(\sum Q^i_\beta(a^*_0)a_{\beta,-1} + c_i a^*_{\alpha_i,-1} + a^*_{\alpha_i,0}b_{i,-1}\big)|0\rangle, z\Big) = \sum_{\beta\in\Phi_+} :Q^i_\beta(a^*(z))a_\beta(z): + c_i\partial_z a^*_{\alpha_i}(z) + a^*_{\alpha_i}(z)b_i(z)$$

And it is possible to check by induction that for all basis elements e_α, f_α of $\mathfrak{g}$ the associated series $e_\alpha(z), f_\alpha(z)$ are of the form $Y(\cdot, z)$.

This remark, combined with lemma 5.5.2 implies the following.

Theorem 5.5.1. *There exists an homomorphism of vertex algebras*

$$V_{\kappa_c}(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0$$

such that

$$e_{i,-1}|0\rangle \mapsto \big(a_{\alpha_i,-1} + \sum_{\beta>\alpha} P^i_\beta(a^*_0)a_{\beta,-1}\big)|0\rangle$$
$$h_{i,-1}|0\rangle \mapsto \big(\sum_{\beta\in\Phi_+} -\beta(h_i)a^*_{\beta,0}a_{\beta,-1} + b_{i,-1}\big)|0\rangle$$
$$f_{i,-1}|0\rangle \mapsto \big(\sum_{\beta\in\Phi_+} Q^i_\beta(a^*_0)a_{\beta,-1} + c_i a^*_{\alpha_i,-1} + a^*_{\alpha_i,0}b_{i,-1}\big)|0\rangle$$

5.5.3 Deforming to other Levels

We present an extension of the previous theorem to an arbitrary level κ. For the proof, we refer to [FBZ04][Theorem 6.2.1]

Theorem 5.5.2. *There exists an homomorphism of vertex algebras*

$$V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}$$

such that

$$\begin{aligned}
e_{i,-1}|0\rangle &\mapsto \big(a_{\alpha_i,-1} + \sum_{\beta>\alpha} P^i_\beta(a^*_0)a_{\beta,-1}\big)|0\rangle \\
h_{i,-1}|0\rangle &\mapsto \big(\sum_{\beta\in\Phi_+} -\beta(h_i)a^*_{\beta,0}a_{\beta,-1} + b_{i,-1}\big)|0\rangle \\
f_{i,-1}|0\rangle &\mapsto \big(\sum_{\beta\in\Phi_+} Q^i_\beta(a^*_0)a_{\beta,-1} + (c_i + (\kappa-\kappa_c)(e_i,f_i))a^*_{\alpha_i,-1} + a^*_{\alpha_i,0}b_{i,-1}\big)|0\rangle
\end{aligned}$$

5.6 Conformal and quasi-conformal structures

We study in this section the conformal properties of the vertex algebras $M_\mathfrak{g}$ and $\pi_0^{\kappa-\kappa_c}$. Let's start by noticing that these vertex algebras may be decomposed as tensor product of simpler algebras.

We start by studying the simplest case, the one of $\mathfrak{sl}_2$. Denote by $M := M_{\mathfrak{sl}_2}$ since there is only one root we drop the subscript α and use the fields $a(z)$, $a^*(z)$ as generating fields for M.

Remark 5.6.1. There is an isomorphism

$$M_\mathfrak{g} \simeq \bigotimes_{\alpha\in\Phi^+} M_\alpha$$

where M_α is the copy of M induced by the immersion

$$M \hookrightarrow M_\mathfrak{g} \qquad \text{which maps the fields} \quad a^*(z)\mapsto a^*_\alpha(z) \quad a(z)\mapsto a_\alpha(z)$$

Proposition 5.6.1. *The $\mathbb{Z}_+$ graded vertex algebra M is conformal with conformal vector*

$$\omega = a_{-1}a^*_{-1}|0\rangle$$

In addition this is the only possible conformal vector.

Proof. It is an easy computation to prove that $a_{-1}a^*_{-1}$ is a conformal vector.

Any other possible choice for a conformal vector must be of the form

$$\big(\lambda a_{-1}a^*_{-1} + \mu a_{-2} + \nu a^*_{-2}\big)|0\rangle$$

But a simple evaluation of the OPEs

$$[a(z), Y(\omega, w)] \qquad [a^*(z), Y(\omega, w)]$$

shows that this coefficients must be exactly

$$\lambda = 1 \qquad \mu = 0 \qquad \nu = 0$$

□

Corollary 5.6.1. *The vertex algebra $M_\mathfrak{g}$ is conformal with conformal vector*

$$\Big(\sum_{\alpha\in\Phi_+} a_{\alpha,-1}a^*_{\alpha,-1}\Big)|0\rangle$$

Analogous statements holds for $\pi_0^{\kappa-\kappa_c}$. We introduce the notation $\pi_0^{\kappa-\kappa_c}(\mathfrak{h})$ to precise the abelian Lie algebra on which is defined the form $\kappa-\kappa_c$ on the definition of π_0 as a vacuum Verma module. Consider an orthogonal basis h_i for which

$$(\kappa-\kappa_c)(h_i,h_j)=\lambda_i\delta_{ij}$$

and consider the subalgebras $\pi_0^{\lambda_i}(h_i)$ generated by the field $b_i(z)$ under the commutation relations $[b_{i,n},b_{i,m}]=n\lambda_i\delta_{n,-m}$.

Remark 5.6.2. There is a natural isomorphism of vertex algebras

$$\pi_0^{\kappa-\kappa_c}(\mathfrak{h})=\bigotimes_{i=1}^{l}\pi_0^{\lambda_i}(h_i)$$

And as conformal structures are concerned we have the following.

Proposition 5.6.2. *The vertex algebra $\pi_0^k(b)$ generated by the field $b(z)$ with commutation relations $[b_n,b_m]=n2k\delta_{n,-m}$ is conformal if and only if $k\neq 0$. In addition it has a $\mathbb{C}$-family of conformal vectors*

$$\omega^{k,\lambda}:=\Big(\frac{1}{4k}b_{-1}b_{-1}-\frac{\lambda}{4k}b_{-2}\Big)|0\rangle \qquad \lambda\in\mathbb{C}$$

These are all possible conformal vectors.

Proof. Note that since π_0^0 is abelian, it cannot be conformal. The rest of the proof is just a simple computation, using the OPE formula. □

5.6.1 Quasi-conformal structure on π_0

Consider the vertex algebra π_0^k for $k\neq 0$ and its conformal vector $\omega^{k,\lambda}$ and write

$$Y(\omega,z)=\sum_{n\in\mathbb{Z}}\omega_n^{k,\lambda}z^{-n-1}=\sum_{n\in\mathbb{Z}}L_nz^{-n-2} \qquad \text{so } L_n=\omega_{n+1}^{k,\lambda}$$

we compute the commutation relations between L_n and b_m

$$[b_n,L_m]=[b_n,\omega_{m+1}]=\sum_{k\geq 0}\binom{n}{k}(b_k\omega^{k,\lambda})_{n+m+1-k}=nb_{n+m}-n(n-1)\lambda\delta_{n+m-2,-1}$$

Proposition 5.6.3. *The conformal action of $\omega^{k,\lambda}$ on $\pi_0^k(b)$ induces through a limit process a quasi conformal structure on the abelian vertex algebra $\pi_0(b)$ given by the above formulas.*

Proof. We will use the definitions given in section 7.4.1 and emulate the proof of Proposition 3.4.3. Consider the $\mathbb{C}[\beta]$ vertex algebra $\pi_0^\beta(b)$, and notice that we may define $\mathbb{C}[\beta]$-linear operators L_n acting on $(\hat{b})^\beta$ with the above formulas. The operators L_n with $n\geq -1$ preserve the Lie subalgebra $\mathbb{C}b[[t]][\beta]\oplus\mathbb{C}[\beta]\mathbf{1}$ and therefore they act also on π_β^0.

A simple calculation provides us with the equation

$$4\beta L_n=\big((b_{-1}b_{-1}-\lambda b_{-2})|0\rangle\big)_{n+1}$$

and therefore they satisfy the commutation relations of the axioms of a quasi conformal vertex algebras. Since π_0^β has no torsion elements the same equations must be satisfied by the $\mathbb{C}[\beta]$ operators L_n themselves, and therefore they satisfy such relations for the specialization at $\beta=0$. □

5.6.2 Free field realization and conformal structures

The above considerations make us wonder if the free field realization

$$V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}(\mathfrak{h})$$

preserves to conformal structures for $\kappa \neq \kappa_c$ or the quasi conformal structures at the critical level. Recall that the conformal vector on $V_\kappa(\mathfrak{g})$ is taken to be

$$S_\kappa := \frac{\kappa_\mathfrak{g}}{2(\kappa-\kappa_c)} \sum_a J^a_{-1} J_{a,-1}|0\rangle$$

Proposition 5.6.4. *Consider $\kappa \neq \kappa_c$ then the conformal vector $S_\kappa \in V_\kappa(\mathfrak{g})$ is sent through the free field realization to the vector*

$$\left(\sum_{\alpha\in\Phi_+} a_{\alpha,-1} a^*_{\alpha,-1}\right)|0\rangle + \frac{\kappa_\mathfrak{g}}{\kappa-\kappa_c}\left(\frac{1}{2}\sum_{i=1}^{l} b_{i,-1} b^i_{-1} - \rho_{-2}\right)|0\rangle$$

in particular the free field realization is a morphism of conformal vertex algebras for $\kappa \neq \kappa_c$.

Since for at the critical value both structures are obtained through the same 'limiting' process the free field realization of $V_{\kappa_c}(\mathfrak{g})$ is of quasi conformal vertex algebras. The structure of quasi-conformal vertex algebra on π_0 is given by the following formulas

$$\begin{aligned} L_n \cdot b_{i,m} &= -m b_{i,m+n} && -1 \leq n < -m \\ L_n \cdot b_{i,-n} &= n(n+1) && n > 0 \\ L_n \cdot b_{i,m} &= 0 && n > -m \end{aligned}$$

Proof. For the proof see [Fre07][Proposition 6.2.2 and Section 6.2.4] □

5.7 Semi-infinite Parabolic induction 1

In this last section of the chapter we generalize some more the results we obtained so far. In particular a slight modification of the proof of theorem 5.5.2 allows us to treat the case of a general parabolic subalgebra $\mathfrak{p}$ (as we will see, so far we only dealt with the case $\mathfrak{p} = \mathfrak{b}_-$).

Consider a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ that contains the lower Borel subalgebra $\mathfrak{b}_-$ (in particular $\mathfrak{h} \subset \mathfrak{p}$. Let

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}$$

be a Levi decomposition of $\mathfrak{p}$. So $\mathfrak{m}$ is a Levi subalgebra containing $\mathfrak{h}$ and $\mathfrak{r}$ is the unipotent radical of $\mathfrak{p}$. In particular let $\Phi_\mathfrak{p}$ the root system of $\mathfrak{p}$, induced by the adjoint action of $\mathfrak{h} \subset \mathfrak{p}$. We naturally have $\Phi_\mathfrak{p} \subset \Phi$. We may take

$$\mathfrak{m} = \bigoplus_{\alpha:\{\alpha,-\alpha\}\subset\Phi_\mathfrak{p}} \mathfrak{g}_\alpha \oplus \mathfrak{h} \qquad \mathfrak{r} = \bigoplus_{\alpha\in\Phi_\mathfrak{p}:-\alpha\notin\Phi_\mathfrak{p}} \mathfrak{g}_\alpha$$

Let

$$\mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i \oplus \mathfrak{h}$$

be the decomposition of $\mathfrak{m}$ into a direct sum of simple Lie algebras $\mathfrak{m}_i, i \geq 1$ and an abelian Lie algebra $\mathfrak{m}_0$ such that these summands are orthogonal to each other with respect to the killing form of $\mathfrak{g}$. Given a set of invariant inner products κ_i on $\mathfrak{m}_i$ for $i = 1, \dots s$, consider the corresponding Kac-Moody algebra $\hat{\mathfrak{m}}_{i,\kappa_i}$ and consider the vacuum Verma module $V_{\kappa_i}(\mathfrak{m}_i)$ for each $i = 1, \dots, s$ and denote by $V_{\kappa_0}(\mathfrak{m}_0)$ the $\hat{\mathfrak{m}}_{0,\kappa_0}$ module $\pi_0^{\kappa_0}$ defined as above. Considering all these vertex algebras together we define

$$V_{(\kappa_i)}(\mathfrak{m}) := \bigotimes_{i=1}^{s} V_{\kappa_i}(\mathfrak{m}_i)$$

with the tensor vertex algebra structure.

Finally consider the Weyl algebra $A^{\mathfrak{g},\mathfrak{p}}$ defined as $A^{\mathfrak{g}}$ but with generators $a^*_{\alpha,n}, a_{\alpha,n}$ for $n \in \mathbb{Z}$ and $\alpha \in \Phi_+ \setminus \Phi_{\mathfrak{p}}$. And as before consider its Fock representation $M_{\mathfrak{g},\mathfrak{p}}$ with its vertex algebra structure.

Notice that in the case in which $\mathfrak{p} = \mathfrak{b}_-$ we have $\mathfrak{m} = \mathfrak{h}$ and $\mathfrak{r} = \mathfrak{n}_+$, in particular $V_{(\kappa_i)}(\mathfrak{m}) = \pi_0^{\kappa}$ and $M_{\mathfrak{g},\mathfrak{p}} = M_{\mathfrak{g}}$. So we are actually considering a more general case. We prove the following analogue of theorem 5.5.2.

Theorem 5.7.1. *Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{m}, \mathfrak{r}$ as above and let κ_i a set of invariant inner products on $\mathfrak{m}_i$ such that there exists an invariant inner product κ on $\mathfrak{g}$ such that*

$$\kappa_i - \kappa_{i,c} = \kappa_{|\mathfrak{m}_i}$$

where $\kappa_{i,c}$ for $i = 1, \dots, s$ is the critical value for the simple Lie algebras $\mathfrak{m}_i$, while $\kappa_{0,c} = 0$. There there exists an homomorphism of vertex algebras

$$w^{\mathfrak{p}}_{\kappa} : V_{\kappa+\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}} \otimes V_{(\kappa_i)}(\mathfrak{m})$$

Proof. See [Fre07, Theorem 6.3.1] □

Chapter 6

Wakimoto modules and applications to the center

We are going to define in this section the so called Wakimoto modules. First we are going to study a little more precisely the morphism

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$$

we will focus in particular on showing that it is actually injective, so this really is a 'realization' of $V_{\kappa_c}(\mathfrak{g})$ in a free field algebra.

Next we are going to define what a module over a vertex algebra is and see the relationship between $V_\kappa(\mathfrak{g})$ modules an $\hat{\mathfrak{g}}_k$ modules.

Finally we will define the Wakimoto modules as $V_{\kappa_c}(\mathfrak{g})$ modules obtained by the pullback of certain $M_{\mathfrak{g}} \otimes \pi_0$ modules and study some of their properties. We identify the Verma module $\mathbb{M}_{0,\kappa_c}$ with a certain Wakimoto module. This characterization allows us to describe better the space of invariants $\zeta(\mathfrak{g}) = V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}$ and ultimately to prove that

$$\operatorname{gr} \zeta(\mathfrak{g}) \simeq \mathbb{C}[J\mathfrak{g}^*]^{\mathfrak{g}[[t]]}$$

6.1 Injectivity

We start by proving that the homomorphism defined in theorem 5.5.2

$$V_\kappa(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$$

is injective for every invariant inner product κ. We start by considering the finite dimensional case.

6.1.1 The finite dimensional case

Recall that in Proposition 5.5.1 we constructed an homomorphism of Lie algebras

$$\mathfrak{g} \to \mathrm{Vect}(N_+ \times H)$$

which actually has image contained in $\mathrm{Vect}(N_+) \oplus \mathbb{C}[N_+] \otimes \mathfrak{h}$ where we embed $\mathfrak{h} \hookrightarrow \mathrm{Vect}(H)$ with the natural homomorphism induced by left multiplication.

We obtain a morphism of associative algebras

$$U(\mathfrak{g}) \to D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*] \subset D(N_+ \times H)$$

where $D(N_+)$ is the Weyl algebra of differential operators on N_+ and $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}\,(\mathfrak{h})$ is embedded in $D(N_+ \times H)$ as the algebra generated by the vector fields in $\mathfrak{h} \subset \mathrm{Vect}(H)$.

Consider now the natural filtration on $D(N_+)$ defined by the order of the differential operator $D(N_+)_{\leq i}$ and the natural filtration on the symmetric algebra $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h})$, we define a filtration on $D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*]$ setting

$$(D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*])_{\leq n} := \bigoplus_{i=0}^{n} D(N_+)_{\leq i} \otimes \mathbb{C}[\mathfrak{h}^*]_{\leq n-i}$$

Since the Lie subalgebra $\mathrm{Vect}(N_+) \oplus \mathbb{C}[N_+] \otimes \mathfrak{h}$ lies in degree ≤ 1. The homomorphism $U(\mathfrak{g}) \to D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*]$ preserves the filtrations. In addition we have equalities

$$\mathrm{gr}\, U(\mathfrak{g}) = \mathrm{Sym}\,\mathfrak{g} \qquad \mathrm{gr}\, D(N_+) = \mathbb{C}[T^*N_+] \qquad \mathrm{gr}\,\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[\mathfrak{h}^*]$$

and

$$\mathrm{gr}\, D(N_+) \otimes \mathbb{C}[\mathfrak{h}^*] = (\mathrm{gr}\, D(N_+)) \otimes (\mathrm{gr}\,\mathbb{C}[\mathfrak{h}^*])$$

Thus we obtain an homomorphism

$$\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*N_+] \otimes \mathbb{C}[\mathfrak{h}^*]$$

Proposition 6.1.1. *Consider the isomorphisms*

$$T^*N_+ = \mathfrak{n}_+^* \times N_+$$

and the identifications $\mathfrak{h} = \mathfrak{h}^*, \mathfrak{n}_- = \mathfrak{n}_+^*, \mathfrak{g} = \mathfrak{g}^*$ *induced by the killing form. Then the morphism of schemes*

$$p : \mathfrak{b}_- \times N_+ \to \mathfrak{g}$$

induced by the homomorphism of algebras $\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*N_+] \otimes \mathbb{C}[\mathfrak{h}^*]$ *and the above identifications, is described by*

$$(x, g) \mapsto gxg^{-1} = \mathrm{Ad}_g(x)$$

in particular its image is open and dense in $\mathfrak{g}$ *and* p *is generically one to one and therefore the homomorphism*

$$\mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*N_+] \otimes \mathbb{C}[\mathfrak{h}^*]$$

is injective.

Proof. By construction the action of $\mathfrak{g}$ is given by a vector bundle morphism which may be written as follows

$$\mathfrak{g} \times (N_+ \times H) \to T(N_+ \times H) \qquad (J^a, x) \mapsto \Big(\sum_\alpha P^a_\alpha(x) e_\alpha + \sum_i Q^a_i(x) b_i, x\Big)$$

where, by the explicit formulas we gave in proposition 5.5.1, $P^a_\alpha, Q^a_i \in \mathbb{C}[N_+]$. The morphism induced on the dual bundles is described as follows:

$$(\mathfrak{n}_+^*\times\mathfrak{h}^*)\times(N_+\times H)\to\mathfrak{g}^*\times(N_+\times H)\quad (e^*_\alpha,x)\mapsto\Big(\sum_a P^a_\alpha(x)(J^a)^*,x\Big)\quad (b^*_i,x)\mapsto\Big(\sum_a Q^a_i(x)(J^a)^*,x\Big)$$

since P^a_α, Q^a_i do not depend on H we find that the morphism

$$(\mathfrak{n}_+^*\times\mathfrak{h}^*)\times N_+\to\mathfrak{g}^*$$

obtained by composing the above morphism with the projection along the first factor is described at the level of function exactly by the morphism $\mathbb{C}[\mathfrak{g}^*]\to\mathbb{C}[T^*N_+]\otimes\mathbb{C}[\mathfrak{h}^*]$ we wanted to describe. Now let $\pi:\mathfrak{g}\to\mathfrak{n}_+\oplus\mathfrak{h}$ the linear projection induced by the decomposition $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$, by definition we have that the morphism $\mathfrak{g}\times(N_+\times H)\to(\mathfrak{n}_+\times\mathfrak{h})\times(N_+\times H)$ is described by

$$(\xi,x)\mapsto(\pi(x\xi x^{-1}),x)$$

The dual morphism $\pi^*:\mathfrak{n}_+^*\times\mathfrak{h}^*\to\mathfrak{g}^*$, after identifying $\mathfrak{n}_+^*=\mathfrak{n}_-$, $\mathfrak{h}^*=\mathfrak{h}$ and $\mathfrak{g}^*=\mathfrak{g}$ through the Killing form is described exactly by the inclusion $\mathfrak{b}_-\to\mathfrak{g}$. It follows that our morphism of interest is described as desired

$$\mathfrak{b}_-\times N_+\to\mathfrak{g}\qquad(\xi,x)\mapsto x\xi x^{-1}$$

□

6.1.2 The vertex algebra case

Consider now the homomorphism of vertex algebras

$$V_\kappa(\mathfrak{g})\to M_\mathfrak{g}\otimes\pi_0^{\kappa-\kappa_c}$$

and consider the filtration on $V_\kappa(\mathfrak{g})$ induced by the PBW filtration, the filtration on $M_\mathfrak{g}$ induced by the filtration on $\widetilde{A}^\mathfrak{g}$ where $\widetilde{A}^\mathfrak{g}_{\leq n}$ are the differential operators of order at most n (i.e. polynomials where in each monomial appear at most n factors of the form $a_{\alpha,m}$) and finally the PBW filtration on $\pi_0^{\kappa-\kappa_c}$ (i.e. the order of a monomial in the $b_{i,n}$).

It is quite clear by the formulas we presented that the homomorphism $V_\kappa(\mathfrak{g})\to M_\mathfrak{g}\otimes\pi_0^{\kappa-\kappa_c}$ preserves these filtrations, and that these are actually vertex algebra filtrations.

In addition is not difficult to prove following equalities

$$\operatorname{gr}V_\kappa(\mathfrak{g})=\mathbb{C}[J\mathfrak{g}^*]\qquad\operatorname{gr}M_\mathfrak{g}=\mathbb{C}[JT^*N_+]\qquad\operatorname{gr}\pi_0^{\kappa-\kappa_c}=\mathbb{C}[J\mathfrak{h}^*]$$

as (commutative) vertex algebras. We obtain an homomorphism of commutative algebras

$$\mathbb{C}[J\mathfrak{g}^*]\to\mathbb{C}[JT^*N_+]\otimes\mathbb{C}[J\mathfrak{h}^*]$$

the following lemma easily follows from the definitions.

Lemma 6.1.1. *The morphism associated to $\mathbb{C}[J\mathfrak{g}^*]\to\mathbb{C}[JT^*N_+]\otimes\mathbb{C}[J\mathfrak{h}^*]$ on the level of schemes*

$$J\mathfrak{h}^*\times JT^*N_+\to J\mathfrak{g}^*$$

is the jet morphism Jp *of the morphism described in the finite dimensional case*

$$p : \mathfrak{h}^* \times T^*N_+ \to \mathfrak{g}^*$$

In particular the map

$$\mathbb{C}[J\mathfrak{g}^*] \to \mathbb{C}[JT^*N_+] \otimes \mathbb{C}[J\mathfrak{h}^*]$$

is injective.

Proof. The first part of the proposition is evident from the definitions. We therefore restrict ourselves to proving that if

$$p : X \to Y$$

is a morphism of schemes isomorphic to $\mathbb{A}^n$ whose image is open and dense in Y and which is generically one to one then the morphism of rings

$$(Jp)^\# : \mathbb{C}[JY] \to \mathbb{C}[JX]$$

is injective. It is quite clear that the finite dimensional morphism $(J_n p)^\#$ is injective, since the image of $J_n p$ is still open and dense in $J_n Y$ and $J_n p$ is still generically one to one. Now any regular function on JY comes from a function on $J_n Y$ (since $Y \simeq \mathbb{A}^m$) and the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[J_n Y] & \xhookrightarrow{(J_n p)^\#} & \mathbb{C}[J_n X] \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[JY] & \xrightarrow[(Jp)^\#]{} & \mathbb{C}[JX]
\end{array}$$

concludes the proof. □

This proposition has the following fundamental corollary.

Theorem 6.1.1. *The homomorphism of vertex algebras*

$$V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}$$

is injective for every κ*.*

Proof. Notice that by construction $\operatorname{gr} V_\kappa(\mathfrak{g}) = \mathbb{C}[J\mathfrak{g}^*]$, $\operatorname{gr} M_\mathfrak{g} = \mathbb{C}[JT^*N_+]$, $\operatorname{gr} \pi_0 = \mathbb{C}[J\mathfrak{h}^*]$ and the induced morphism $\mathbb{C}[J\mathfrak{g}^*] \to \mathbb{C}[JT^*N_+] \otimes \mathbb{C}[J\mathfrak{h}^*]$ is exactly $(Jp)^\#$. Then the theorem follows from the following general fact.

If V and W are two filtered vector spaces and $\varphi : V \to W$ is a linear map preserving the filtrations such that the associated graded map

$$\operatorname{gr} \varphi : \operatorname{gr} V \to \operatorname{gr} W$$

is injective, then φ is injective.

To see this consider an a vector $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$ and let $n \in \mathbb{Z}_+$ such that $v \in V_{\leq n} \setminus V_{\leq n-1}$. Then we have

$$\operatorname{gr} \varphi(\mathrm{Symb}\, v) = \mathrm{Symb}\, \varphi(v) = 0$$

But by hypothesis $\operatorname{gr} \varphi$ is injective so $\mathrm{Symb}\, v = 0$ which implies $v = 0$.

□

6.2 The center and π_0

We prove the following

Proposition 6.2.1. *The center $\zeta(\mathfrak{g})$ is mapped through the free field realization embedding*

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$$

into the subspace π_0. In particular the embedding above induces an embedding of abelian vertex algebras

$$\zeta(\mathfrak{g}) \hookrightarrow \pi_0$$

Proof. Notice that any element in the center maps to a $\mathfrak{g}[[t]]$ invariant element of $M_{\mathfrak{g}} \otimes \pi_0$ thus we only need to show that

$$(M_{\mathfrak{g}} \otimes \pi_0)^{\mathfrak{g}[[t]]} \subset \pi_0$$

One may prove by induction that the lexicographically ordered monomials

$$\prod_{l_a<0} b_{i_a,l_a} \prod_{m_b<0} e^R_{\beta_b,m_b} \prod_{n_c\leq 0} a^*_{\alpha_c,n_c}|0\rangle$$

form a basis of $M_{\mathfrak{g}} \otimes \pi_0$. Write this decomposition as

$$M_{\mathfrak{g}} \otimes \pi_0 = \overline{W}_{0,\kappa_c} \otimes M_{\mathfrak{g},+}$$

a proof completely analogous to the one of Proposition 5.1.2 proves that $M_{\mathfrak{g},+}$ is isomorphic, as an $\mathfrak{n}_+[[t]]$-module to $U(\mathfrak{n}_+[[t]])^\vee$ and therefore its space of $\mathfrak{n}_+[[t]]$ invariants is one dimensional. Since the action of $\mathfrak{n}_+[[t]]$ commutes with the operators $b_{i,n}, e^R_{\beta,m}$ we find that a $\mathfrak{g}[[t]]$ invariant vector must belong to $\overline{W}_{0,\kappa_c}$. Finally a $\mathfrak{g}[[t]]$ invariant element must also be $\mathfrak{h}$ invariant, in particular its weight must be 0 and hence no factors $e^R_{\beta,m}$ can occur. This concludes the proof. □

6.3 Modules over a vertex algebra

Definition 6.3.1. A **module** over a vertex algebra V is a vector space M equipped with a linear map

$$Y^M : V \otimes M \to M((z))$$

or equivalently a linear map

$$Y^M(\cdot, z) : V \to \mathrm{End}(M)[[z^{\pm 1}]]$$

with image contained in the subspace of fields. Such that $Y^M(|0\rangle, z) = \mathrm{id}_M$ and for any elements $A, B \in V$, $m \in M$ the formal power series

$$Y^M(A,z)Y^M(B,w)m \qquad Y^M(B,w)Y^M(A,z)m \qquad Y^M(Y(A,z-w)B,w)m$$

are expressions of the same element in

$$M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

in the three corresponding spaces

$$M((z))((w)) \qquad M((w))((z)) \qquad M((w))((z-w))$$

We list a couple of remarks without giving any proof. The facts we list if not trivial are easily proven emulating proofs of the structure theory for vertex algebras a more detailed discussion may be found in [FBZ04][Chapter 5]. We see that a lot of the equalities typical of vertex algebras are still valid in the context of modules.

- There is a natural notion of morphism of modules. By definition a morphism of V modules $\varphi : M_1 \to M_2$ is a linear map which satisfies

$$\varphi(A_n^{M_1} \cdot v) = A_n^{M_2} \cdot \varphi(v)$$

- We have the equality

$$Y^M(TA, z) = \partial_z Y^M(A, z)$$

  this easily follows from the fact that $TA = A_{-2}|0\rangle$ and from the equality (which follows from the axioms taking $B = |0\rangle$)

$$Y^M(A, z) = \sum_{n \geq 0} Y^M(A_{-n-1}|0\rangle, w)(z-w)^n$$

- It immediately follows from the axioms that if we denote by A_k^M the k-th coefficient of $Y^M(A, z)$

$$[A_k^M, B_l^M] = \sum_{n \geq 0} \binom{k}{n} (A_n B)_{k+l-n}^M$$

- The usual vertex operator

$$Y : V \to \mathrm{End}\,(V)[[z^{\pm 1}]]$$

  defines a structure of V module on V itself.

- Given a W module M and an homomorphism $\varphi : V \to W$ of vertex algebras, M gains a natural structure of V module with the module structure defined by $Y_V^M = Y_W^M \circ \varphi$;

- Modules has natural compatible structures with respect to tensor product. If M is a V module and N is a W module then $M \otimes N$ is naturally a $V \otimes W$ module with

$$Y^{M \otimes N} = Y^M \otimes Y^N$$

  with the obvious meaning.

- Given a V module M the linear map defined by

$$U(V) \to \mathrm{End}\,(M) \qquad A_{[n]} \mapsto A_n^M$$

  is a Lie algebra homomorphism.

6.3.1 V modules and $\widetilde{U}(V)$ modules

We prove here that the notion of a module over a vertex algebra V is equivalent to the notion of a module over the complete associative algebra $\widetilde{U}(V)$ introduced in Definition 4.2.3.

Theorem 6.3.1. *Any module over a vertex algebra V is naturally a continuous module over the complete algebra $\widetilde{U}(V)$. Vice versa, any continuous module over $\widetilde{U}(V)$ is naturally a V module. The category of continuous $\widetilde{U}(V)$ modules and the category of V module are equivalent.*

Proof. See [FBZ04][Theorem 5.1.6] □

In particular we see that for any Lie algebra $\mathfrak{g}$ with an associative symmetric form κ. The category of $V_\kappa(\mathfrak{g})$ modules and the category of $\widetilde{U}_\kappa(\hat{\mathfrak{g}})$ module coincide. In particular modules over $V_\kappa(\mathfrak{g})$ and smooth $\hat{\mathfrak{g}}_\kappa$ modules coincide.

We restate this in the following two sections, providing some more explicit descriptions.

π_0 modules

We describe here the modules over the commutative vertex algebra π_0, we will see that the notion of a module over a vertex algebra and module over the associated commutative algebra slightly differ.

Proposition 6.3.1. *To give a module over π_0 is equivalent to give a smooth module over the abelian Lie algebra $\hat{\mathfrak{h}}$.*

Proof. To give a π_0 module M consists simply in defining fields

$$b_i(z) \in \mathrm{End}\,(M)[[z^{\pm 1}]]$$

such that all their Fourier coefficients commute. This is exactly like the construction of a smooth $\hat{\mathfrak{h}}$ module sending

$$b_{i,n} \leftrightarrow h_{i,n}$$

the condition of smoothness and field coincide. □

For instance consider any character $\chi : h \to \mathbb{C}$, consider a one dimensional representation $\mathbb{C}_\chi$ of the Lie subalgebra $\mathfrak{h}[[t]] \oplus \mathbb{C}\mathbf{1}$ of $\hat{\mathfrak{h}}$ where $\mathbf{1}$ acts as the identity, $t\mathfrak{h}[[t]]$ acts like 0 while $\mathfrak{h}$ acts as the multiplication by χ. Define

$$\pi_\chi := \mathrm{Ind}\,_{\mathfrak{h}[[t]]\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{h}}}\mathbb{C}_\chi$$

It is a naturally a smooth $\hat{\mathfrak{h}}$ module and hence a π_0 module.

$V_\kappa(\mathfrak{g})$ modules and $\hat{\mathfrak{g}}_\kappa$ modules

We explore here the relationship between $\hat{\mathfrak{g}}_\kappa$ modules and $V_\kappa(\mathfrak{g})$ modules.

Proposition 6.3.2. *To give a module over $V_\kappa(\mathfrak{g})$ is equivalent to give a smooth module over the affine Kac-Moody algebra $\hat{\mathfrak{g}}_\kappa$.*

Proof. Consider a $V_\kappa(\mathfrak{g})$ module M. Then the linear map

$$\hat{\mathfrak{g}}_k \to \mathrm{End}(M) \qquad J_n^a \mapsto Y^M(J_{-1}^a|0\rangle, z)_n$$

is well defined (i.e. it extends from $\mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}\mathbf{1}$ to $\mathfrak{g}((t)) \oplus \mathbb{C}\mathbf{1}$) by the hypothesis that Y^M has image contained in the subspace of fields. And a proof similar to lemma 5.5.2 shows that it is an homomorphism of Lie algebras.

On the other hand given a smooth $\hat{\mathfrak{g}}_\kappa$ module M define a structure of $V_\kappa(\mathfrak{g})$ by setting

$$Y^M(J_{-1}^a|0\rangle, z) := \sum_{n\in\mathbb{Z}} J_n^a z^{-n-1}$$

where we interpret the J_n^a as endomorphisms of M. And as always

$$Y^M(J_{n_1}^{a_1} \dots J_{n_m}^{a_m}|0\rangle, z) := \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} : \partial_z^{-n_1-1} J_{n_1}^{a_1} \dots \partial_z^{-n_m-1} J^{a_m}(z) :$$

emulating a proof of theorem 4.2.1 one can prove that this actually defines a structure of $V_\kappa(\mathfrak{g})$ module on M. □

6.4 Wakimoto modules and applications to the center

We are ready to define the Wakimoto modules.

Definition 6.4.1. Let M be a $M_\mathfrak{g}$ module and N a π_0 module. We call $M \otimes N$ the Wakimoto module associated to M and N. It is a $\hat{\mathfrak{g}}_{\kappa_c}$ smooth module with the structure induced by the tensor structure on $M \otimes N$ and the homomorphism

$$V_{\kappa_c}(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0$$

We will focus on the study of the module W^+_{0,κ_c}, which we will soon define, since it is the most important for our purposes.

Let $A^\mathfrak{g}_{++}$ be the subalgebra of $A^\mathfrak{g}$ generated by the elements $a_{\alpha,n}, a^*_{\alpha,m}$ with $n > 0$ and $m \geq 0$. Consider its trivial one dimensional module $\mathbb{C}|0\rangle'$ where all the $a_{\alpha,n}, a^*_{\alpha,m}$ act like 0 while 1 acts as the identity consider the induced representation

$$M'_\mathfrak{g} := \mathrm{Ind}\,^{A^\mathfrak{g}}_{A^\mathfrak{g}_{++}} \mathbb{C}|0\rangle'$$

it has a basis of monomials of the form

$$a^*_{\alpha_1,n_1} \dots a^*_{\alpha_k,n_k} a_{\alpha_1,m_1} \dots a_{\alpha_k,m_l}|0\rangle' \qquad n_i < 0, m_j \leq 0$$

it is naturally an $\widetilde{A}^\mathfrak{g}$ module and a $M_\mathfrak{g}$ module.

Next consider $\rho : \mathfrak{h} \to \mathbb{C}$ to be the sum of the fundamental weights $\omega_i : \mathfrak{h} \to \mathbb{C}$. Consider $\pi_{-2\rho}$ as a π_0 module.

We define the $\hat{\mathfrak{g}}_{\kappa_c}$ module

$$W^+_{0,\kappa_c} := M'_\mathfrak{g} \otimes \pi_{-2\rho}$$

it has a basis of monomials of the following form

$$b_{i_1,r_1}\dots b_{i_s,r_s}a^*_{\alpha_1,n_1}\dots a^*_{\alpha_k,n_k}a_{\beta_1,m_1}\dots a_{\beta_l,m_l} \qquad n_i,r_i<0, m_j\le 0$$

obtained from the structure of $M_{\mathfrak{g}}\otimes\pi_0$ module and precomposing with the involution of $\hat{\mathfrak{g}}_{\kappa_c}$ defined by $e_{i,n}\mapsto f_{i,n}, h_{i,n}\mapsto -h_{i,n}$, in particular the action of $\hat{\mathfrak{g}}_{\kappa_c}$ is described by the following formulas

$$\begin{aligned} e_i(z) &\mapsto \sum_{\beta\in\Phi_+} :Q^i_\beta(a^*(z))a_\beta(z): + c_i\partial_z a^*_{\alpha_i}(z) + a^*_{\alpha_i}(z)b_i(z) \\ h_i(z) &\mapsto \sum_{\beta\in\Phi_+}\beta(h_i) : a^*_\beta(z)a_\beta(z) : - b_i(z) \\ f_i(z) &\mapsto a_{\alpha_i}(z) + \sum_{\beta>\alpha_i} :P^i_\beta(a^*(z))a_\beta(z): \end{aligned}$$

Given a character $\lambda:\mathfrak{h}\to\mathbb{C}$ consider the Verma module

$$\mathbb{M}_{\lambda,\kappa_c} := \mathrm{Ind}^{\hat{\mathfrak{g}}_{\kappa_c}}_{\hat{\mathfrak{n}}_+\oplus\mathfrak{h}\oplus\mathbb{C}\mathbf{1}}\mathbb{C}_\lambda$$

where $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+\oplus t\mathfrak{g}[[t]]$ and $\mathbb{C}_\lambda$ is the $\hat{\mathfrak{n}}_+\oplus\mathfrak{h}\oplus\mathbb{C}\mathbf{1}$ module where $\hat{\mathfrak{n}}_+$ acts trivially, $\mathfrak{h}$ acts according to λ and $\mathbf{1}$ acts like the identity. These Verma modules are the analogue of the classical Verma modules for a finite dimensional simple Lie algebra $\mathfrak{g}$. If the affine case $\hat{\mathfrak{n}}_+$ plays the role of the upper nilpotent subalgebra while $\mathfrak{h}$ plays the role of the maximal toral subalgebra. These representations are well studied and essential in the representation theory of the affine algebra $\hat{\mathfrak{g}}_{\kappa_c}$.

We will need the following theorem which characterizes the weights of irreducible subquotients of Verma modules. It can be found in [KK79, Theorem 2].

Theorem 6.4.1 (Kac-Kazhdan)**.** *A weight (μ,n) appears as the highest weight of an irreducible subquotient of the Verma module $\mathbb{M}_{\hat{\lambda}}$ where $\hat{\lambda}=(\lambda,0)$ if and only if $n\le 0$ and either $\lambda=\mu$ or there exists a sequence of weights $\mu_0,\dots,\mu_m$ such that $\mu_0=\lambda$ and $\mu_{i+1}=\mu_i\pm m_i\beta_i$ for some positive roots β_i and positive integers m_i which satisfy*

$$2(\mu_i+\rho,\beta_i) = m_i(\beta_i,\beta_i)$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form.

In other words the weight (μ,n) appears as the highest weight of an irreducible subquotient of $\mathbb{M}_{\hat{\lambda}}$ if and only if $n\le 0$ and there exists an element w in the Weyl group of $\mathfrak{g}$ such that

$$\mu = w(\lambda+\rho)-\rho$$

Proposition 6.4.1. *The Wakimoto module W^+_{0,κ_c} is isomorphic to the Verma module $\mathbb{M}_{0,\kappa_c}$.*

Proof. The vector $|0\rangle'\otimes|-2\rho\rangle$ satisfy , by direct computation

$$\hat{\mathfrak{n}}_+\oplus\mathfrak{h}\cdot\big(|0\rangle'\otimes|-2\rho\rangle\big)=0 \qquad \mathbf{1}\cdot|0\rangle'\otimes|-2\rho\rangle = |0\rangle'\otimes|-2\rho\rangle$$

by the properties of induced module we obtain a $\hat{\mathfrak{g}}_{\kappa_c}$-linear map

$$\mathbb{M}_{0,\kappa_c}\to W^+_{0,\kappa_c}$$

Next notice that both modules are actually modules for the extended algebra $\hat{\mathfrak{g}}_{\kappa_c} \rtimes \mathbb{C}L_0$ where $L_0 = -t\partial_t$. In addition a basis for $\mathbb{M}_{0,\kappa_c}$ is given by the PBW theorem:

$$h_{i_1,r_1} \dots h_{i_s,r_s} e_{\alpha_1,n_1} \dots e_{\alpha_k,n_k} f_{\beta_1,m_1} \dots f_{\beta_l,m_l} \qquad n_i, r_i < 0, m_j \leq 0$$

since for any $h \in \mathfrak{h}$ we have $h \cdot a^*_{\alpha,n} = \alpha(h) a^*_{\alpha,n}$ and $h \cdot a_{\alpha,n} = -\alpha(h) a_{\alpha,n}$ while $L_0 \cdot a^*\alpha, n = -n a^*_{\alpha,n}$ and $L_0 \cdot a_{\alpha,n} = -n a_{\alpha,n}$, we find that the characters of $\mathbb{M}_{0,\kappa_c}$ and W^+_{0,κ_c} under the action of $\mathfrak{h} \oplus \mathbb{C}L_0$ are equal. To show that the above morphism is an isomorphism its enough to show that it is surjective, or equivalently that W^+_{0,κ_c} is generated by $|0\rangle' \otimes |-2\rho\rangle$ as a $\hat{\mathfrak{g}}_{\kappa_c}$-module.

Suppose by contradiction that W^+_{0,κ_c} is not generated by $|0\rangle' \otimes |-2\rho\rangle$. Consider the Lie subalgebra $\hat{\mathfrak{n}}_- := \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$. We first claim that

$$W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c}$$

has a non zero weight component of degree ≤ 0. Indeed consider its $\hat{\mathfrak{g}}_{\kappa_c}$ submodule

$$W' = U(\hat{\mathfrak{g}}_{\kappa_c})|0\rangle \otimes |-2\rho\rangle \subsetneq W^+_{0,\kappa_c}$$

And consider an homogeneous vector $w \in W^+_{0,\kappa_c} \setminus W'$ of maximal weight. Then $w \notin \hat{\mathfrak{n}}_- W^+_{0,\kappa_c}$. Suppose by contradiction that this is false, them all the homogeneous factors w_i in an expression of

$$w = \sum_i x_i w_i \qquad \text{with } x_i \in \hat{\mathfrak{n}}_-$$

must be of a weight higher than the weight of w and therefore must belong to W' then since W' is a submodule the same must be true for w so we deduce that $w \in W'$. This contradiction proves that $w \notin \hat{\mathfrak{n}}_- W^+_{0,\kappa_c}$. In particular, since w was taken to be homogeneous, there is a non zero weight component in $W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c}$. Since W^+_{0,κ_c} is the direct sum of its weight components which are finite dimensional, the same must be true for its quotient $W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c}$ in particular writing

$$W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c} = \bigoplus_{(\gamma,n)} \left(W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c} \right)_{(\gamma,n)}$$

we find that there is a non zero weight (γ, n) such that $\left(W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c} \right)_{(\gamma,n)} \neq 0$ and considering any non zero linear functional

$$\left(W^+_{0,\kappa_c} / \hat{\mathfrak{n}}_- W^+_{0,\kappa_c} \right)_{(\gamma,n)} \to \mathbb{C}$$

we get a non zero linear $\hat{\mathfrak{n}}_-$-invariant functional $\varphi : W^+_{0,\kappa_C} \to \mathbb{C}$ of weight different from 0. Next notice that there is a basis of W^+_{0,κ_c} formed by the ordered monomials

$$h_{i_1,r_1} \dots h_{i_s,r_s} f_{\alpha_1,n_1} \dots f_{\alpha_k,n_k} a^*_{\alpha_1,m_1} \dots a^*_{\alpha_l,m_l} |0\rangle' \otimes |-2\rho\rangle \tag{6.1}$$

this follows from the explicit formulas above, noticing that the change of coordinates from the classical one to this one is triangular. In particular the action of $L_-\mathfrak{b}_- = t^{-1}\mathfrak{b}_-[t^{-1}]$ is free. A $\hat{\mathfrak{n}}_-$ invariant linear functional must be also $L_-\mathfrak{b}_-$-invariant and using the basis introduced above we easily see that the space of coinvariants with respect to $L_-\mathfrak{b}_-$ is isomorphic to the subspace spanned by the monomials

$$a^*_{\alpha_1,m_1} \dots a^*_{\alpha_l,m_l} |0\rangle' \otimes |-2\rho\rangle$$

The induced functional $\varphi : W^+_{0,\kappa_c}/L_-\mathfrak{b}_-W^+_{0,\kappa_c} \to \mathbb{C}$ is by hypothesis non zero and homogeneous. Since it must be nonzero on some homogeneous vector we find that the weight of φ must be of the form

$$-\sum_j (-\beta_j, n_j) \qquad \text{with } \beta_j > 0, n_j < 0$$

in addition since by construction the weight of φ is not 0 this sum is not empty.

Now φ induces a $\hat{\mathfrak{n}}_-$-invariant functional on $\mathbb{M}_{0,\kappa_c}$ or in other words a lowest weight vector

$$\varphi \in \mathbb{M}^\vee_{0,\kappa_c}$$

the weight of lowest weight vectors on $\mathbb{M}^\vee_{0,\kappa_c}$ are similarly described as in [KK79] above. So we must have

$$0 \neq \sum_i \beta_i = w(\rho) - \rho$$

for some w in the Weyl group, and since for any $w \in W$ the element $w(\rho) - \rho$ is a sum of negative simple roots $\sum_i \beta_i$ cannot be of this form. This contradiction concludes the proof. □

This characterization has incredibly useful. Using nice basis as in 6.1 we can characterize the space of $\widetilde{\mathfrak{b}}_+ := \mathfrak{b}_+ \oplus t\mathfrak{g}[[t]]$ invariants of the Verma Module $\mathbb{M}_{0,\kappa_c}$

Lemma 6.4.1. *The space of $\widetilde{\mathfrak{b}_+}$-invariants of W^+_{0,κ_c} is equal to the space $\pi_{-\rho} \subset W^+_{0,\kappa_c}$*

Proof. Consider the operators $f^R_{\alpha,n}$ introduced in Proposition 5.4.4. we call them $f^R_{\alpha,n}$ instead of $e^R_{\alpha,n}$ because we are considering an action of $\hat{\mathfrak{g}}_{\kappa_c}$ obtained by the free field realization pre-composed with the involution.

Analogously to 6.1 we find that W^+_{0,κ_c} has a basis of the form

$$b_{i_1,r_1}\dots b_{i_s,r_s} f^R_{\alpha_1,n_1}\dots f^R_{\alpha_k,n_k} a^*_{\beta_1,m_1}\dots a^*_{\beta_l,m_l}|0\rangle' \otimes |-2\rho\rangle$$

We write for convenience

$$b_{[i],[r]} f^R_{[\alpha],[n]} a^*_{[\beta][m]}$$

where, for instance, $[n]$ stands for the ordered vector $(n_1,\dots,n_k)$. Consider now a $\widetilde{\mathfrak{b}}_+$ invariant vector v and write it as

$$v = \sum_{([i],[r]),([\alpha],[n])} b_{[i],[r]} f^R_{[\alpha],[n]} v^*_{[i],[r],[\alpha],[n]}$$

with $v^*_{[i],[r],[\alpha],[n]} \in Span(a^*_{[\beta],[m]})$. Such an expression is unique. Since v in $\widetilde{\mathfrak{b}}_+$ invariant it is in particular $L_+\mathfrak{n}_- = t\mathfrak{n}_-[[t]]$ invariant. Since the action of $L_+\mathfrak{n}_-$ commutes with the action of the $b_{i,n}$ (by the formulas) and with the action of $f^R_{\alpha,m}$ by construction we find (by uniqueness of the above expression) that

$$L_-\mathfrak{n}_- v^*_{[i],[r],[\alpha],[n]]} = 0$$

We can prove analogously to Proposition 5.1.2 that as an $L_+\mathfrak{n}_-$ module $W^{+,*}_{0,\kappa_c} = Span(a^*_{[\beta],[m]})$ is isomorphic to

$$U(L_+\mathfrak{n}_-)^\vee$$

and therefore its space of invariants is one dimensional and spanned by 1. This proves that a $\widetilde{\mathfrak{b}}_+$ invariant vector must be of the form

$$v = \sum_{([i],[r]),([\alpha],[n])} b_{[i],[r]} f^R_{[\alpha],[n]}$$

But it must also be annihilated by the Lie subalgebra $\mathfrak{h} \subset \widetilde{\mathfrak{b}}_+$. Recall that by the remarks on the right action we have

$$[h_0, f^R_{\alpha,n}] = \alpha(h)$$

we find that v must be of the form

$$v = \sum_{([i],[r])} b_{[i],[r]}$$

that is to say $v \in \pi_{-2\rho}$. This proves

$$(W^+_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} \subset \pi_{-2\rho}$$

Finally since $\widetilde{\mathfrak{b}}_+$ kills the vacuum vector and commutes with the action of $b_{i,n}$ we find that also

$$\pi_{-2\rho} \subset (W^+_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+}$$

□

6.5 Semi-infinite parabolic induction 2

With the language of modules we are now ready to use theorem 5.7.1 to construct what we will call the semi-infinite parabolic induction functor.

Consider a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and let $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ be a Levi decomposition, $\mathfrak{m} = \oplus_{i=1}^{s} \mathfrak{m}_i \oplus \mathfrak{m}_0$ a decomposition of $\mathfrak{m}$ where the $\mathfrak{m}_i$ are simple Lie algebras for $i = 1, \dots, s$, $\mathfrak{m}_0$ is abelian and all the factors are orthogonal with respect to the killing form on $\mathfrak{g}$. Let κ_i a set of inner products on $\mathfrak{m}_i$. We may consider the Lie algebra $\hat{\mathfrak{m}}_{(\kappa_i)}$ and construct as usual its vacuum module $V_{(\kappa_i)}(\mathfrak{m})$. In the same fashion of proposition 6.3.2 there is a correspondence between smooth $\hat{\mathfrak{m}}_{(\kappa_i)}$ modules and $V_{(\kappa_i)}(\mathfrak{m})$ modules.

We have the following corollary of theorem 5.7.1.

Corollary 6.5.1. *For any smooth $\hat{\mathfrak{m}}_{(\kappa_i)}$-module R, where the inner products κ_i satisfy the conditions of theorem 5.7.1, the tensor product*

$$M_{\mathfrak{g},\mathfrak{p}} \otimes R$$

is naturally a $V_{\kappa+\kappa_c}(\mathfrak{g})$ module and hence a smooth $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$ module.

In addition to every $\hat{\mathfrak{m}}_{(\kappa_i)}$-morphism $f : R_1 \to R_2$ the induced morphism

$$id \otimes f : M_{\mathfrak{g},\mathfrak{p}} \otimes R_1 \to M_{\mathfrak{g},\mathfrak{p}} \otimes R_2$$

is an homomorphism of $V_{\kappa+\kappa_c}(\mathfrak{g})$ modules and hence of $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$-modules.

Thus we obtain a functor from the category of smooth $\hat{\mathfrak{m}}_{(\kappa_i)}$ modules to the category of smooth $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$ modules. We call this functor the **semi-infinite parabolic induction functor**. And we will call the smooth $\hat{\mathfrak{g}}_{\kappa+\kappa_c}$ module $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ the **generalized Wakimoto module corresponding to** R.

6.6 Invariants

The goal of this section is to prove that

$$\mathrm{gr}\,\zeta(\mathfrak{g}) = \mathrm{Inv}\ J\mathfrak{g}^*$$

we will extensively use the facts we studied for the Wakimoto module W^{+}_{0,κ_c}.

In order to prove this statement we compare the $\hat{\mathfrak{g}}_{\kappa_c}$ module $V_{\kappa_c}(\mathfrak{g})$ to the Verma module $\mathbb{M}_{0,\kappa_c}$.

The vacuum vector $|0\rangle \in V_{\kappa_c}(\mathfrak{g})$ is killed by the Lie subalgebra $\widetilde{\mathfrak{b}_+} = \hat{\mathfrak{n}}_+ \oplus (\mathfrak{h} \otimes 1)$ and $\mathbf{1}$ acts like the identity on it. Therefore we obtain a homomorphism of $\hat{\mathfrak{g}}_{\kappa_c}$ modules

$$\mathbb{M}_{0,\kappa_c} \to V_{\kappa_c}(\mathfrak{g})$$

induced by the embedding $\mathbb{C}_0 \to \mathbb{C}|0\rangle \to V_{\kappa_c}(\mathfrak{g})$ which is an homomorphism of $\widetilde{\mathfrak{b}_+}$ modules by the above remarks. This is of course surjective (apply the PBW theorem).

Remark 6.6.1. The following equality between spaces of invariants hold

$$V_{\kappa_c}(\mathfrak{g})^{\widetilde{\mathfrak{b}_+}} = V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]} \qquad \mathrm{gr}\left(V_{\kappa_c}(\mathfrak{g})\right)^{\widetilde{\mathfrak{b}_+}} = \mathrm{gr}\left(V_{\kappa_c}(\mathfrak{g})\right)^{\mathfrak{g}[[t]]}$$

Indeed we only need to show that every $\widetilde{\mathfrak{b}_+}$ invariant vector is also $\mathfrak{g}$ invariant. This is true because $V_{\kappa_c}(\mathfrak{g})$ and $\mathrm{gr}\, V_{\kappa_c}(\mathfrak{g})$ are both direct sum of finite dimensional representations of $\mathfrak{g}$. For a finite dimensional representation V of $\mathfrak{g}$ we have

$$V^{\mathfrak{g}} = V^{\mathfrak{b}_+}$$

and this is enough to show the remark.

To proceed towards our goal of showing that $\mathrm{gr}\,\zeta_{\kappa_c}(\hat{\mathfrak{g}}) = \mathrm{Inv}\ J\mathfrak{g}^*$ we consider the following diagram

$$\begin{array}{ccc}
\mathrm{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}_+}}\right) & \longrightarrow & \mathrm{gr}\left(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}\right) \\
\big\downarrow & & \big\downarrow \\
\mathrm{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}_+}} & \longrightarrow & \mathrm{gr}\left(V_{\kappa_c}(\mathfrak{g})\right)^{\mathfrak{g}[[t]]}
\end{array}$$

We will show that the left vertical arrow is an isomorphism while the lower horizontal arrow is surjective. This easily implies that the right vertical arrow is surjective as well, since we already knew it is an embedding we obtain the sought after equality

$$\mathrm{gr}\left(\zeta_{\kappa_c}(\mathfrak{g})\right) = \mathrm{Inv}\ J\mathfrak{g}^*$$

we start by describing the space $\mathrm{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}_+}}$.

Proposition 6.6.1. *The graded space* $\mathrm{gr}\,(\mathbb{M}_{0,\kappa_c})$ *is isomorphic, as a* $\widetilde{\mathfrak{b}_+}$ *module to the space of functions on the closed subscheme*

$$J\mathfrak{g}^*_{(-1)} \subset L_{-1}\mathfrak{g}^*$$

defined by the equations $J^a_0 = 0$ *for* $J^a \in \mathfrak{n}_+ \oplus \mathfrak{h}$.

In addition the space of $\widetilde{\mathfrak{b}_+}$ *invariant functions on this closed subscheme is equal to*

$$\mathbb{C}[P_{i,m_i}]_{i=1,\dots,l;m_i<d_i}$$

Proof. By the PBW theorem

$$\mathrm{gr}\,\mathbb{M}_{0,\kappa_c} = \mathrm{Sym}\left(\mathfrak{g}((t))/(t\mathfrak{g}[[t]] \oplus \mathfrak{b}_+)\right) = \mathbb{C}[J\mathfrak{g}^*_{(-1)}]$$

is addition it is clear that the subspace $J\mathfrak{g}^*(-1) \subset L_{-1}\mathfrak{g}^*$ is preserved by the action of $\widetilde{\mathfrak{b}}_+$ and the isomorphism above is $\widetilde{\mathfrak{b}}_+$ invariant.

Using the isomorphism

$$(\cdot t) : L_{-1}\mathfrak{g}* \to J\mathfrak{g}^*$$

we restrict ourselves to compute the space of invariants of the closed subscheme

$$J\mathfrak{g}^*_{(0)} \to J\mathfrak{g}^* \qquad J\mathfrak{g}^*_{(0)} = \left\{J^a_{-1} = 0 : J^a \in \mathfrak{n}_+ \oplus \mathfrak{h}\right\}$$

Consider the $\widetilde{B}_+$, the subgroup of JG defined as

$$\widetilde{B}_+(R) = \left\{g \in G(R[[t]]) : g(0) \in B(R)\right\}$$

It is an affine closed subgroup of JG, its Lie algebra is then isomorphic to $\mathfrak{b}_+ \oplus t\mathfrak{g}[[t]] = \widetilde{\mathfrak{b}}_+$. We also consider its finite dimensional analogues

$$\widetilde{B}_{+,n} \subset J_nG \qquad \widetilde{B}_{+,n}(R) = \left\{g \in G(R[t]/t^n) : g(0) \in B(R)\right\}$$

These are all closed subgroups of J_nG. Recall that in Lemma 4.5.1 we proved that the map

$$J_n p : J_n(\mathfrak{g}^*_{reg}) \to J_n\mathcal{P}$$

is a geometric quotient for every $n \geq 0$. Our first goal to deduce from this that the map

$$\{J^a_{-1} = 0 : J^a \in \mathfrak{b}_+\} =: J_n(\mathfrak{g}^*_{reg})_{(0)} \to J_n\mathcal{P}_{(0)} := \{P_{i,-1} = 0\}$$

is well defined and a geometric quotient for the action of $\widetilde{B}_{+,n}$. Applying [MFK94][Proposition 0.2] to prove that this is a geometric quotient it suffices to prove that the map is surjective on $\mathbb{C}$ points and that the $\mathbb{C}$ fibers are single $\widetilde{B}_{+,n}(\mathbb{C})$ orbits.

We will describe these maps using the isomorphism $\mathfrak{g}^* = \mathfrak{g}$ induced by the Killing form. Notice that since $\mathfrak{g}_{reg}$ is an open subset of $\mathfrak{g}$ we have

$$J_n(\mathfrak{g}_{reg})(R) = \mathfrak{g}_{reg}(R) + t\frac{\mathfrak{g}[t]}{t^n} \otimes R$$

this follows from the fact that for an open subscheme $U \subset X$ we have the equality $U(R[t]/t^n) = \{x \in X(R[t]/t^n : x(0) \in U(R)\}$. Call $\mathfrak{n}_{+,reg} := \mathfrak{n}_+ \cap \mathfrak{g}_{reg}$, we naturally have

$$J_n(\mathfrak{g}^*_{reg})_{(0)} = \mathfrak{n}_{+,reg}(R) \oplus t\frac{\mathfrak{g}[t]}{t^n} \otimes R$$

Now to see that the restriction

$$J_n(\mathfrak{g}^*_{reg})_{(0)} \to J_n\mathcal{P}_{(0)}$$

By classical results about the space of invariants we have that the ideal generated by the $\mathfrak{g}$ invariant polynomials in $\mathfrak{g}_{reg}$ is the ideal of functions vanishing on the nilpotent cone. Thus every nilpotent element lies is the set of zeroes of the invariant polynomials so for every $x \in J_n(\mathfrak{g}^*_{reg})_{(0)}$ we deduce $P_{i,-1}(x) = P_i(x(0)) = 0$ and the restriction is well defined.

To see surjectivity consider a point $u \in J_n\mathcal{P}_{(0)}$ and a point $x \in J_{\mathfrak{g}_{reg}}$ over it. By assumption $P_i(x(0)) = 0$ for every i so $x(0)$ is nilpotent and regular and therefore there exists a $g \in G(\mathbb{C})$ such that $(g \cdot x)(0) \in \mathfrak{n}_+$ or in other words $g \cdot x \in J_n(\mathfrak{g}_{reg})_{(0)}$ and since $J_n p$ is G invariant $g \cdot x \mapsto u$.

Finally to see that the fibers are single $\widetilde{B}_{+,n}$ orbits consider two points $x, y \in J_n(\mathfrak{g}_{reg})_{(0)}$ such that their image in $J_n\mathcal{P}$ is the same. Since the map $J_n(\mathfrak{g}_{reg}) \to J_n\mathcal{P}$ has fibers consisting of single $J_nG(\mathbb{C})$ orbits we know that there exists an element $g \in J_nG(\mathbb{C})$ such that $g \cdot x = y$. In particular both $x(0)$ and $(g \cdot x)(0) = g(0) \cdot x(0)$ lie in $\mathfrak{n}_{+,reg}$. Since there is only one Borel subalgebra $\mathfrak{b}_x$ such that $x(0) \in \mathfrak{b}_x$ and $g(0) \cdot x(0) \in \mathfrak{b}_{gx}$ we must have $\mathfrak{b}_x = \mathfrak{b}_+$ and $\mathfrak{b}_{gx} = \mathfrak{b}_+ = \mathfrak{g}(0)\mathfrak{b}_+$ so $g(0)$ must normalize $\mathfrak{b}_+$ and therefore it must lie in B_+. This proves that $\mathfrak{g} \in \widetilde{B}_{+,n}$.

We are now able to deduce that since $J_n p : J_n(\mathfrak{g}^*_{reg})_{(0)} \to J_n\mathcal{P}_{(0)}$ is a geometric quotient we have the following equality

$$\mathbb{C}[J_n(\mathfrak{g}^*_{reg})_{(0)}]^{\mathfrak{b}\oplus t\mathfrak{g}[t]/t^n} = \mathbb{C}[J_n\mathcal{P}_{(0)}]$$

From this we deduce that the natural map

$$\mathbb{C}[P_{i,n}]_{n<-1} \to \mathbb{C}[J\mathfrak{g}^*_{(0)}]^{\widetilde{\mathfrak{b}}_+}$$

is an isomorphism.

To see injectivity is enough to notice that for every $n \geq 0$ the composition

$$\mathbb{C}[P_{i,n}]_{n<-1} \to \mathbb{C}[J\mathfrak{g}^*_{(0)}]^{\widetilde{\mathfrak{b}}_+} \to \mathbb{C}[J_n(\mathfrak{g}^*_{reg})_{(0)}]^{\mathfrak{b}\oplus t\mathfrak{g}[t]/t^n} = \mathbb{C}[J_n\mathcal{P}_{(0)}]$$

Is exactly the quotient of $\mathbb{C}[P_{i,n}]$ by the ideal generated by $(P_{i,m})_{m<-n-1}$. To see surjectivity it suffice to show that every $\widetilde{\mathfrak{b}}_+$ invariant function on $J(\mathfrak{g}^*_{reg})_{(0)}$ come from a $\mathfrak{b} \oplus t\mathfrak{g}[t]/t^n$ invariant function on $J_n(\mathfrak{g}^*_{reg})_{(0)}$. This can be done emulating the end of the proof of Theorem 4.5.2. □

As an immediate corollary we obtain that the map

$$\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}_+}} \to \operatorname{gr}\left(V_{\kappa_c}(\mathfrak{g})\right)^{\mathfrak{g}[[t]]}$$

is surjective.

In addition we can easily compute the character of under the L_0 action of both $\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}_+}}\right) = \operatorname{gr}\left(\pi_{-2\rho}\right)$ and $\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}_+}}$ and see that they are actually equal, so the immersion

$$\operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}^{\widetilde{\mathfrak{b}_+}}\right) \hookrightarrow \operatorname{gr}\left(\mathbb{M}_{0,\kappa_c}\right)^{\widetilde{\mathfrak{b}_+}}$$

is actually an isomorphism. Combining these statements we obtain

Proposition 6.6.2. *The following equality between graded spaces holds*

$$\operatorname{gr} \zeta(\mathfrak{g}) = \operatorname{Inv} J\mathfrak{g}^*$$

in particular the character of $\operatorname{gr} \zeta(\mathfrak{g})$ *and hence also the character of* $\zeta(\mathfrak{g})$ *under the action of* L_0 *is given by the following formula*

$$\operatorname{ch} \zeta(\mathfrak{g}) = \prod_{i=1}^{l} \prod_{n_i \geq d_i + 1} \frac{1}{1 - q^{n_i}}$$

Chapter 7

Screening Operators

7.1 Overview

The goal of this chapter is the construction of certain operators

$$\overline{S}_i : W_{0,\kappa_c} \to \widetilde{W}^{(i)}_{0,0,\kappa_c}$$

where $W_{0,\kappa_c} = M_{\mathfrak{g}} \otimes \pi_0$ and $\widetilde{W}^{(i)}_{0,0,\kappa_c}$ are certain $\hat{\mathfrak{g}}_{\kappa_c}$ modules that we are going to define. These are called the **screening operators** (or screening operators of the second kind, following [Fre07, Chapter 7.3]): they intertwine with the action of $\hat{\mathfrak{g}}_{\kappa_c}$ and annihilate the vacuum vector in $M_{\mathfrak{g}} \otimes \pi_0 = W_{0,\kappa_c}$.

It follows that the vertex subalgebra $V_{\kappa_c}(\mathfrak{g})$ is contained in the intersection of the kernels of these operators. In particular since the center $\zeta(\hat{\mathfrak{g}})$ is contained in the abelian subalgebra π_0 it is also contained the intersection of the kernels of the operators

$$\overline{V}_i[1] := (\overline{S}_i)_{|\pi_0}$$

We are going to write down explicit formulas for the screening operators and therefore for the operators $\overline{V}_i[1]$ as well.

In the next chapter we will give a geometric interpretation of the operators $\overline{V}_i[1]$, this will finally allow us to identify the center $\zeta(\hat{\mathfrak{g}})$ with the algebra of functions on the space of ${}^L G$ Opers on the formal disc D.

7.2 Intertwining Operators

We start by giving the definition of a particular kind of **intertwining operators** in the context of a conformal vertex algebra V and a V-module M.

Remark 7.2.1. Consider a conformal vertex algebra V of central charge c with conformal vector ω. Then any V-module M carries an action of the Virasoro algebra of central charge c through the operators

$$L_n := \omega^M_n \quad \text{where} \quad Y^M(\omega, z) = \sum_{n\in\mathbb{Z}} \omega^M_n z^{-n-2}$$

We call $T := \omega^M_{-1} \in \mathrm{End}\,(M)$

Proof. The commutation relations of the ω^M_n are easily described by the axioms of a module. □

Definition 7.2.1. Let V be a conformal vertex algebra and let $T \in \mathrm{End}\,(M)$ defined as above. We define a linear map

$$Y_{V,M} : M \to \mathrm{Hom}(V, M)[[z^{\pm 1}]] \qquad Y_{V,M}(A, z)B := e^{zT} Y_M(B, -z)A$$

this is an example of an **intertwining operator**.

The fundamental properties of $Y_{V,M}$ are described in the following proposition.

Proposition 7.2.1. *Let V be a confomal vertex algebra, M a V-module and $Y_{V,M}$ as above. Then given any $A, B \in V$ and any $B \in M$ there exists an element*

$$f \in M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$

such that

$$\begin{array}{rl} Y_M(A, z)Y_{V,M}(B, w)C & \quad Y_{V,M}(B, w)Y(A, z)C \\ Y_{V,M}(Y_{V,M}(B, w - z)A, z)C & \quad Y_{V,M}(Y_M(A, z - w)B, w)C \end{array}$$

are expansions of f in

$$M((z))((w)) \quad M((w))((z)) \quad M((z))((w - z)) \quad M((w))((z - w))$$

respectively.

Proof. See [FHL93, Proposition 5.1.2] □

Corollary 7.2.1. *Let $V, M, Y_{V,M}$ as above. The following hold:*

$$Y_M(A, z)Y_{V,M}(B, w) - Y_{V,M}(B, w)Y(A, z) = \sum_{k \geq 0} \frac{1}{k!} Y_{V,M}(A^M_k B, w)\partial^k_w \delta(z - w)$$

In particular

$$A^M_n \circ B^{V,M}_m - B^{V,M}_m \circ A_n = \sum_{k \geq 0} \binom{n}{k} (A^M_k B)^{V,M}_{n+m-k} \tag{7.1}$$

Proof. Analogous to the proof of corollary 3.2.2. □

We will write the above formula, abusing notation as

$$[A_n, B_m] = \sum_{k \geq 0} \binom{n}{k} (A_k B)_{n+m-k}$$

7.2.1 Example: intertwining operators for π_0^β

To be more concise we write $\beta := k+2 \in \mathbb{C}$ in what follows we are interested in the cases for which $k \neq k_c$, so in terms of β we are assuming that $\beta \neq 0$. As a useful example we describe an intertwining operator in the case of the conformal vertex algebra π_0^β for $\beta \neq 0$ and its module $\pi_{2\beta}^\beta$.

Recall that π_0^β is the vacuum module associated to the one dimensional abelian Lie algebra spanned by an element b with the symmetric form given by $\kappa(b,b) = 2\beta$. So the elements b_n satisfy the relations

$$[b_n, b_m] = n2\beta\delta_{n,-m}$$

Its conformal vector is $\omega = \frac{1}{4\beta} b_{-1}b_{-1}|0\rangle - \frac{1}{2\beta} b_{-2}$ and the translation operator is therefore given by $T = \frac{1}{4\beta}\sum_{n\in\mathbb{Z}} b_n b_{-n-1}$.

We want to compute the intertwining operator

$$V_{2\beta}(z) \in \mathrm{Hom}(\pi_0^\beta, \pi_{2\beta}^\beta)[[z^{\pm 1}]] \qquad V_{2\beta}(z) := Y_{\pi_0^\beta, \pi_{2\beta}^\beta}(|2\beta\rangle, z)$$

Note that by formula 7.1 we have

$$[b_n, V_{2\beta}(z)_m] = \sum_{k\geq 0} \binom{n}{k} (b_k|2\beta\rangle)_{n+m-k}^{\pi_0^\beta, \pi_{2\beta}^\beta} = 2\beta V_{2\beta}(z)_{n+m}$$

or more compactly

$$[b_n, V_{2\beta}(z)] = 2\beta z^n V_{2\beta}(z)$$

Since the action of the operators $b_n : n < 0$ generates π_0^β the commutation relations above together with the datum $V_{2\beta}(z)|0\rangle$ uniquely determines $V_{2\beta}(z)$.

Proposition 7.2.2. *The operator $V_{2\beta}(z)$ is described by the following formula.*

$$V_{2\beta}(z) = T_{2\beta} \exp\Big(\sum_{n<0} -\frac{b_n}{n} z^{-n}\Big) \exp\Big(\sum_{n>0} -\frac{b_n}{n} z^{-n}\Big)$$

where $T_{2\beta} : \pi_0^\beta \to \pi_{2\beta}^\beta$ is the operator commuting with all $b_n : n < 0$ that sends $|0\rangle \mapsto |2\beta\rangle$.

Proof. Call $e(z)$ the formula of the right hand side. By the remarks above, to show that $e(z) = V_{2\beta}(z)$ it suffices to show that

$$e(z)|0\rangle = V_{2\beta}(z)|0\rangle \quad \text{and} \quad [b_n, e(z)] = 2\beta z^n e(z)$$

To show the first equality recall that by definition $V_{2\beta}(z)|0\rangle = e^{zT}|2\beta\rangle$. And notice that

$$e(z)|0\rangle = T_{2\beta} \exp\Big(-\sum_{n<0} \frac{b_n}{n} z^{-n}\Big)|0\rangle$$

Since it is clear that $e(z)|0\rangle = |2\beta\rangle + z(\dots)$ we can restrict ourselves to prove that

$$Te(z)|0\rangle = \partial_z e(z)|0\rangle = T_{2\beta}\big(\sum_{n<0} b_n z^{-n-1}\big) \exp\Big(-\sum_{n<0} \frac{b_n}{n} z^{-n}\Big)|0\rangle$$

In order to do this notice that by definition we have $[b_0, T_{2\beta}] = 2\beta T_{2\beta}$ and $[b_m, T_{2\beta}] = 0$ for every $m \neq 0$ and therefore $[T, T_{2\beta}] = T_{2\beta} b_{-1}$ in addition notice that

$$\left[T, -\sum_{n<0} \frac{b_n}{n} z^{-n}\right] = \sum_{n<0} b_{n-1} z^{-n} = \sum_{n<-1} b_n z^{-n-1}$$

It easily follows that

$$\begin{aligned} Te(z)|0\rangle &= [T, T_{2\beta}] \exp\Big(-\sum_{n<0} \frac{b_n}{n} z^{-n}\Big)|0\rangle + T_{2\beta}\Big[T, \exp\Big(-\sum_{n<0} \frac{b_n}{n} z^{-n}\Big)\Big]|0\rangle \\ &= T_{2\beta}\Big(b_{-1}\exp(\dots) + \big(\sum_{n<-1} b_n z^{-n-1}\big)\exp(\dots)\Big)|0\rangle = \partial_z\big(e(z)|0\rangle\big) \end{aligned}$$

We are left to compute the commutation relations $[b_n, e(z)]$. We find that

- ($n = 0$) The operator b_0 commutes with the 'exponential parts' while $[b_0, T_{2\beta}] = 2\beta T_{2\beta}$ so we have
$$[b_0, e(z)] = 2\beta e(z)$$
- ($n < 0$) The operator $b_n : n < 0$ commutes with $T_{2\beta}$ and with the first exponential while
$$\left[b_n, -\sum_{n>0} \frac{b_m}{m} z^{-m}\right] = 2\beta z^n$$
and we finally find
$$[b_n, e(z)] = 2\beta z^n e(z)$$
- ($n > 0$) The same reasoning of the previous point applies to the case $n > 0$ and we get
$$[b_n, e(z)] = 2\beta z^n e(z)$$

This concludes the proof. □

7.3 The $\mathfrak{sl}_2$ case

We start by working out the $\mathfrak{sl}_2$ case. This is a crucial for us since the operators $\overline{S}_i$ are actually obtained with the semi-infinite parabolic induction functor from the $\mathfrak{sl}_2$ case.

First we need to define a new vertex algebra.

7.3.1 Friedan-Martinec-Shenker bosonization

Consider the Lie algebra with generators p_n, q_m with $n, m \in \mathbb{Z}$ and a central element $\mathbf{1}$ with the following commutation relations

$$[p_n, p_m] = n\delta_{n,-m}\mathbf{1} \qquad [q_n, q_m] = -n\delta_{n,-m}\mathbf{1} \qquad [p_n, q_m] = 0$$

For $\lambda, \mu \in \mathbb{C}$ let $\Pi_{\lambda,\mu}$ be the induced representation from the Lie subalgebra spanned by $p_m, q_n, \mathbf{1}$ with $n, m \geq 0$ of the one dimensional module generated by a vector $|\lambda, \mu\rangle$ where

$$p_n|\lambda,\mu\rangle = \lambda\delta_{n,0}|\lambda,\mu\rangle \qquad q_m|\lambda,\mu\rangle = \mu\delta_{m,0}|\lambda,\mu\rangle \qquad \mathbf{1}|\lambda,\mu\rangle = |\lambda,\mu\rangle$$

For $\lambda, \mu, \lambda', \mu'$ such that $\lambda\lambda' - \mu\mu' \in \mathbb{Z}$ consider the vertex operators

$$V_{\lambda,\mu}(z) \in \mathrm{Hom}(\Pi_{\lambda',\mu'}, \Pi_{\lambda'+\lambda,\mu'+\mu})[[z^{\pm 1}]]$$

defined by the formulas

$$V_{\lambda,\mu}(z) := T_{\lambda,\mu} z^{\lambda\lambda'-\mu\mu'} \exp\Big(-\sum_{n<0} \frac{\lambda p_n + \mu q_n}{n} z^{-n}\Big) \exp\Big(-\sum_{n>0} \frac{\lambda p_n + \mu q_n}{n} z^{-n}\Big)$$

Using a little bit of an abuse of notation we will write $e^{\lambda u + \mu v}$ for the operator $V_{\lambda,\mu}(z)$. Here u should be understood as the antiderivative of $p(z)$ while v should be understood as the antiderivative of $q(z)$. For any $\gamma \in \mathbb{C}$ let

$$\Pi_\gamma := \bigoplus_{n\in\mathbb{Z}} \Pi_{n+\gamma,n+\gamma}$$

we define the structure of a vertex algebra on Π_0 with the vertex operators $V_{\lambda,\mu}(z)$. We define

$$Y(|n,n\rangle), z) := V_{n,n}(z) : \Pi_0 \to \Pi_0$$

and check that the fields $Y(|n,n\rangle), b(z)$ are mutually local and satisfy all the axioms of the reconstruction theorem. Every Π_γ becomes naturally a Π_0 module.

Let $M := M_{\mathfrak{sl}_2}$ then we have an embedding.

Theorem 7.3.1. *There is a (unique) embedding of vertex algebras $M \hookrightarrow \Pi_0$ such that the fields $a(z)$ and $a^*(z)$ are mapped to the fields*

$$\tilde{a}(z) = e^{u+v} \qquad \tilde{a}^*(z) = - : p(z) e^{-u-v} :$$

We consider the following modules:

Definition 7.3.1. Let $\gamma, \lambda \in \mathbb{C}$ we define

$$\widetilde{W}_{\gamma,\lambda,k} := \Pi_\gamma \otimes \pi^\beta_\lambda$$

this is naturally a $\Pi_0 \otimes \pi^\beta_0 = \widetilde{W}_{0,0,k}$-module, hence a $M \otimes \pi^\beta_0$-module and hence a $\widehat{\mathfrak{sl}}_{2\,k}$-module

7.3.2 Intertwining operators

Definition 7.3.2. Define the intertwining operator

$$\widetilde{S}_k(z) := Y^{\widetilde{W}_{0,0,k}, \widetilde{W}_{-\beta,2\beta,k}}(|-\beta\rangle \otimes |2\beta\rangle, z) = \tilde{a}(z)^{-\beta} V_{2\beta}(z)$$

We justify this notation with the following remark.

Remark 7.3.1. It is possible to prove, analogously to Proposition 7.2.2 that the intertwining operator associated to $|-\beta, -\beta\rangle \in \Pi_{-\beta}$ is exactly

$$V_{-\beta,-\beta}(z) = T_{-\beta,-\beta} \exp\left(\sum_{n<0} \frac{\beta p_n + \beta q_n}{n} z^{-n}\right) \exp\left(\sum_{n>0} \frac{\beta p_n + \beta q_n}{n} z^{-n}\right)$$

Finally, we follow the notation $\widetilde{a}(z) = e^{u+v}$ and denote y $\widetilde{a}(z)^{-\beta} := e^{-\beta u - \beta v} = V_{-\beta,-\beta}(z)$

We denote by $v_{-\beta,2\beta} = |-\beta\rangle \otimes |2\beta\rangle$ for simplicity.

Lemma 7.3.1. *The following OPEs hold*

$$[e(z), \widetilde{S}_k(w)] = 0$$
$$[h(z), \widetilde{S}_k(w)] = 0$$
$$[f(z), \widetilde{S}_k(w)] = \frac{Y(\beta T v_{-\beta-1,2\beta}, w)}{(z-w)} + \frac{Y(\beta v_{-\beta-1,2\beta}, w)}{(z-w)^2}$$

Proof. By corollary 7.2.1 we know that for any $X \in \mathfrak{sl}_2$

$$[X(z), \widetilde{S}_k(w)] = \sum_{n \geq 0} \frac{Y(X_n v_{-\beta,2\beta}, w)}{(z-w)^{n+1}}$$

Therefore to compute the desired OPE we need to compute the polar parts (in z) of $X(z)v_{-\beta,2\beta}$ for $X \in \mathfrak{g}$. Recall that the morphism $V_k(\mathfrak{sl}_2) \to M \otimes \pi_0^\beta$ maps

$$e(z) \mapsto a(z)$$
$$h(z) \mapsto -2 : a^*(z)a(z) : + b(z)$$
$$f(z) \mapsto - : a^*(z)^2 a(z) : + k\partial_z a^*(z) + a^*(z)b(z)$$

The corresponding fields on $\Pi_0 \otimes \pi_0^\beta$ are given therefore by

$$e(z) \mapsto \widetilde{a}(z)$$
$$h(z) \mapsto -2 : \widetilde{a}^*(z)\widetilde{a}(z) : + b(z)$$
$$f(z) \mapsto - : \widetilde{a}^*(z)^2 \widetilde{a}(z) : + (\beta - 2)\partial_z \widetilde{a}^*(z) + \widetilde{a}^*(z)b(z)$$

where, as before

$$\widetilde{a}(z) = e^{u+v} \qquad \widetilde{a}^*(z) = - : p(z)e^{-u-v} :$$

We proceed with our calculations by steps, paying specific attention on the polar parts of the series we are calculating. As the notation is concerned we write

$$U((z)) \ni a(z) = \sum_{i=-N}^{n} a_i z^i + O(z^{n+1})$$

if $a(z) - \sum_{i=-N}^{n} a_i z^i \in z^{n+1}U[[z]]$ for any vector space U.

1. Notice that $v_{-\beta,2\beta}$ is killed by all $p_n, q_n, b_n : n > 0$ while we have

$$p_0 v_{-\beta,2\beta} = q_0 v_{-\beta,2\beta} = -\beta v_{-\beta,2\beta}$$
$$b_0 v_{-\beta,2\beta} = 2\beta v_{-\beta,2\beta}$$

2. We start by computing $e^{nu+nv} v_{-\beta,2\beta}$:

$$e^{nu+nv} v_{-\beta,2\beta} = v_{-\beta+n,2\beta} + z(np_{-1} + nq_{-1}) v_{-\beta+n,2\beta} + O(z^2)$$

Indeed

$$\begin{aligned} T_{n,n} \exp\Big(-\sum_{m<0} n \frac{p_m + q_m}{m} z^{-m}\Big) \exp\Big(-\sum_{m>0} n \frac{p_m + q_m}{m} z^{-m}\Big) v_{-\beta,2\beta} \\ = T_{n,n} \exp\Big(-\sum_{m<0} n \frac{p_m + q_m}{m} z^{-m}\Big) v_{-\beta,2\beta} \\ = v_{-\beta+n,2\beta} + z(np_{-1} + nq_{-1}) v_{-\beta+n,2\beta} + O(z^2) \end{aligned}$$

Where we used the fact that $v_{-\beta,2\beta}$ is killed by all p_n, q_n with $n > 0$.

3. From point 2 we immediately deduce that

$$e(z)_- v_{-\beta,2\beta} = 0$$

so the first OPE is proved;

4. We consider now $\tilde{a}^*(z)$, we claim that:

$$\tilde{a}^*(z) v_{-\beta,2\beta} = \frac{\beta}{z} v_{-\beta-1,2\beta} + \big(-\beta(p_{-1} + q_{-1}) - p_{-1}\big) v_{-\beta-1,2\beta} + O(z)$$

Indeed

$$\begin{aligned} \tilde{a}^*(z) v_{-\beta,2\beta} = \big(-p(z)_+ e^{-u-v} - e^{-u-v} p(z)_-\big) v_{-\beta,2\beta} = \\ -p_{-1} v_{-\beta-1,2\beta} + O(z) - e^{-u-v} \frac{p_0}{z} v_{-\beta,2\beta} = \\ -p_{-1} v_{-\beta-1,2\beta} + O(z) - e^{-u-v} \frac{-\beta}{z} v_{-\beta,2\beta} \end{aligned}$$

and we conclude using the expression of $e^{-u-v} v_{-\beta,2\beta}$ written in point 2;

5. We are now ready to compute the OPE relative to $h(z)$ and we claim that once again

$$h(z)_- v_{-\beta,2\beta} = 0$$

$$h(z) v_{-\beta,2\beta} = -2\big(\tilde{a}^*(z)_+ \tilde{a}(z) + \tilde{a}(z)\tilde{a}^*(z)_-\big) v_{-\beta,2\beta} + b(z) v_{-\beta,2\beta}$$

We compute these two terms separately

(a)

$$\begin{aligned} -2\big(\tilde{a}^*(z)_+ \tilde{a}(z) + \tilde{a}(z)\tilde{a}^*(z)_-\big) v_{-\beta,2\beta} &= -2\tilde{a}(z)\tilde{a}^*(z)_- v_{-\beta,2\beta} + O(1) \\ &= -2\frac{\beta}{z} v_{-\beta,2\beta} + O(1) \end{aligned}$$

(b)

$$b(z)v_{-\beta,2\beta} = \frac{2\beta}{z}v_{-\beta,2\beta} + O(1)$$

Their sum is clearly regular;

6. To compute $f(z)v_{-\beta,2\beta}$ we need to compute three separate terms

(a) $(:\widetilde{a}^*(z)^2\widetilde{a}(z) : v_{-\beta,2\beta})$ Note first that

$$:\widetilde{a}^*(z)^2\widetilde{a}(z) := \widetilde{a}^*(z)_+^2\widetilde{a}(z) + 2\widetilde{a}^*(z)_+\widetilde{a}(z)\widetilde{a}^*(z)_- + \widetilde{a}(z)\widetilde{a}^*(z)_-^2$$

So using the properties we already proved so far we have

$$-:\widetilde{a}^*(z)^2\widetilde{a}(z) : v_{-\beta,2\beta} = O(1) - 2\widetilde{a}^*(z)_+\widetilde{a}(z)\frac{\beta}{z}v_{-\beta-1,2\beta} - \widetilde{a}(z)\frac{\beta(\beta+1)}{z^2}v_{-\beta-2,2\beta}$$
$$= \frac{2\beta}{z}(p_{-1} + \beta(p_{-1} + q_{-1}))v_{-\beta-1,2\beta} - \frac{\beta(\beta+1)(p_{-1}+q_{-1})}{z}v_{-\beta-1,2\beta}\frac{\beta(\beta+1)}{z^2}v_{-\beta-1,2\beta} + O(1)$$

(b) Knowing $\widetilde{a}^*(z)_-v_{-\beta,2\beta}$ we easily find that

$$(\beta-2)\partial_z\widetilde{a}^*(z)v_{-\beta,2\beta} = -\frac{(\beta-2)\beta}{z^2}v_{-\beta-1,2\beta}$$

(c) Finally

$$\widetilde{a}^*(z)b(z)v_{-\beta,2\beta} = \frac{2\beta^2}{z^2}v_{-\beta-,2\beta} + \frac{1}{z}\big(-2\beta(p_{-1} + \beta(p_{-1}+q_{-1})) + \beta b_{-1}\big)v_{-\beta-1,2\beta}$$

Summing all these contributes together we find

$$f(z)v_{-\beta,2\beta} = \frac{\beta\big(-(\beta+1)(p_{-1}+q_{-1}) + b_{-1}\big)}{z}v_{-\beta-1,2\beta} + \frac{\beta}{z^2}v_{-\beta-1,2\beta}$$

By analogous calculations we find that the coefficient of z^{-1} is exactly $\beta T v_{-\beta-1,2\beta}$ therefore we proved the desired formula for the OPE $[f(z), S_k(w)]$.

□

Corollary 7.3.1. *The residue*

$$\overline{S}_k := \int S_k(w)dw$$

intertwines with the action of $\widehat{\mathfrak{sl}}_{2k}$ *on both modules.*

Proof. By the properties of $Y^{\widetilde{W}_{0,0,k},\widetilde{W}_{-\beta,2\beta,k}}$ we have

$$Y(\beta T v_{-\beta-1,2\beta}, w) = \partial_w Y(\beta v_{-\beta-1,2\beta}, w)$$

So the OPE relative to $f(z)$ may be written as

$$[f(z), S_k(w)] = \partial_w\left(\frac{Y(\beta v_{-\beta-1,2\beta}, w)}{(z-w)}\right)$$

hence its residue is 0 □

7.4 Intertwining operators at the critical level

Our goal is to define an operator analogous to $\overline{S}_k$ when $k = k_c$. We will construct it as the 'limit' of the operators $\overline{S}_k$ for $k \to k_c$. To construct this limit we will slightly modify the definition of all vertex algebras involved in order to make them $\mathbb{C}[\beta]$-vertex algebras, where β is now an indeterminate, such that their specialization to $\beta = k$ brings back the usual vertex algebras of level k.

To construct the limit of the intertwining operators we will need the following lemma.

Lemma 7.4.1. *Let M, N be (not necessarily finite dimensional) free $\mathbb{C}[x]$-modules. And let $\varphi : M \to N$ be an homomorphism of $\mathbb{C}[x]$-modules.*

Then if the specialization $\varphi_\lambda : M/(x-\lambda) \to N/(x-\lambda)$ is 0 for infinitely many $\lambda \in \mathbb{C}$ then φ is identically 0.

Proof. Let $(m_i)_{i\in I}, (n_j)_{j\in J}$ be basis for M and N respectively. In addition consider polynomials $P_i^j(x) \in \mathbb{C}[x]$ such that

$$\varphi(m_i) = \sum_{j\in J} P_i^j(x) n_j$$

note that for a fixed index i only a finite number of P_i^j as j varies in not 0.

Now after we take the specialization at $x = \lambda$ the elements m_i form a $\mathbb{C}$ basis for $M/(x-\lambda)$, and the elements n_j form a $\mathbb{C}$ basis for $N/(x-\lambda)$. The homomorphism φ_λ is easily described as

$$\varphi_\lambda(m_i) = \sum_{j\in J} P_i^j(\lambda) n_j$$

Therefor we find that by the hypothesis of φ_λ being 0 for infinitely many λ for every couple (i,j) the polynomial $P_i^j(x)$ has infinitely many zeros (namely the complex numbers $\lambda \in \mathbb{C}$ for which $\varphi_\lambda = 0$) and is therefore 0. □

7.4.1 $\mathbb{C}[\beta]$ vertex algebras

We define here $\mathbb{C}[\beta]$-vertex algebra analogues of the vertex algebras $V_\kappa(\mathfrak{g})$ and π_0^β. From now on β has to be intended as a variable and **not** as the complex number $k+2$ as in the previous section.

Let $\mathfrak{g}$ a Lie algebra equipped with an associative symmetric form κ consider the $\mathbb{C}[\beta]$-Lie algebra $L\mathfrak{g}[\beta] = L\mathfrak{g} \otimes \mathbb{C}[\beta]$. Consider its one dimensional extension

$$\hat{\mathfrak{g}}_{\beta\kappa} := L\mathfrak{g}[\beta] \oplus \mathbb{C}[\beta]\mathbf{1}$$

with the bracket defined $\mathbb{C}[\beta]$ linearly by the formula

$$[X \otimes f(t) \otimes p(\beta), Y \otimes g(t) \otimes q(\beta)] := p(\beta)q(\beta)\left([X,Y] \otimes f(t)g(t) - \beta\kappa(X,Y)\int f(t)g'(t)dt\right)$$

Since the bracket is $\mathbb{C}[\beta]$-linear we may consider the specialization of $\hat{\mathfrak{g}}_\beta$ at $\beta = k$. The result is clearly the affine algebra $\hat{\mathfrak{g}}_{k\kappa}$. When $\mathfrak{g}$ is simple an $\kappa = \kappa_\mathfrak{g}$ we call this Lie algebra simply $\hat{\mathfrak{g}}_\beta$.

We move on defining the associated vertex algebra. Consider the $\mathbb{C}[\beta]$-subalgebra $\mathfrak{g}[[t]][\beta] \oplus \mathbb{C}[\beta]\mathbf{1} \subset \hat{\mathfrak{g}}_{\beta\kappa}$. and its trivial module $\mathbb{C}[\beta]|0\rangle$, where as always $\mathbf{1}$ acts as the identity. Define

$$V_{\beta\kappa}(\mathfrak{g}) := \operatorname{Ind}_{\mathfrak{g}[[t]][\beta]\oplus\mathbb{C}[\beta]\mathbf{1}}^{\hat{\mathfrak{g}}_{\beta\kappa}} \mathbb{C}[\beta]|0\rangle$$

by the PBW theorem it has a $\mathbb{C}[\beta]$ basis of lexicographically ordered monomials of the form

$$J^{a_1}_{n_1} \dots J^{a_m}_{n_m}|0\rangle$$

defining the vertex operators $\mathbb{C}[\beta]$-linearly as in the case of $V_\kappa(\mathfrak{g})$ defines a structure of $\mathbb{C}[\beta]$ vertex algebra on $V_{\beta\kappa}(\mathfrak{g})$. We denote

$$V_\beta(\mathfrak{sl}_2) \quad \text{and} \quad \pi_0^\beta$$

the vertex algebras built with the above construction from $\mathfrak{sl}_2$ with the normalized killing form $1/4\kappa_{\mathfrak{sl}_2}$ and of the one dimensional abelian Lie algebra spanned by b with the form $\kappa(b,b) = 2$.

Proposition 7.4.1. *There exists an homomorphism of $\mathbb{C}[\beta]$-vertex algebras*

$$V_\beta(\mathfrak{sl}_2) \to M \otimes \pi_0^\beta$$

such that

$$\begin{aligned}
&e_{-1}|0\rangle \mapsto a_{-1}|0\rangle \\
&h_{-1}|0\rangle \mapsto \big(-2a_0^* a_{-1} + b_{-1}\big)|0\rangle \\
&f_{-1}|0\rangle \mapsto \big(-(a_0^*)^2 a_{-1} + (\beta-2)a_{-1}^* + a_0^* b_{-1}\big)|0\rangle
\end{aligned}$$

Proof. Note first that Lemma 5.5.2 applies in the $\mathbb{C}[\beta]$-case as well. Thus we only need to show that the Fourier coefficients of the operators associated to $e(z), h(z), f(z)$ satisfy the commutation relations of $\hat{\mathfrak{sl}}_{2\beta}$. Since they do satisfy this commutation relations on the specialization at $\beta = k+2$ for any $k \in \mathbb{C}$ by lemma 7.4.1 they must satisfy them also in $\operatorname{End}_{\mathbb{C}[\beta]}(M \otimes \pi_0^\beta)$. □

7.4.2 $\mathbb{C}[\beta]$ version of $\overline{S}_k$

Consider the free $\mathbb{C}[\beta]$ modules

$$M \otimes_{\mathbb{C}} \pi_0^\beta \qquad \Pi_0 \otimes_{\mathbb{C}} \pi_0^\beta$$

they have a natural structure of $\mathbb{C}[\beta]$ vertex algebras induced by the structure of $\mathbb{C}$-vertex algebra on M and Π_0 and from the structure of $\mathbb{C}[\beta]$-vertex algebra on π_0^β. Their specialization to $\beta = k+2$ are the vertex algebras $W_{0,k}$ and $\widetilde{W}_{0,0,k}$ respectively. In addition we have embeddings

$$V_\beta(\mathfrak{sl}_2) \hookrightarrow M \otimes_{\mathbb{C}} \pi_0^\beta \hookrightarrow \Pi_0 \otimes_{\mathbb{C}} \pi_0^\beta$$

Next consider the free $\mathbb{C}[\beta]$-module $\Pi_{-\beta+n,-\beta+n}$ with a basis of monomial of the form

$$p_{n_1} \dots p_{n_k} q_{m_1} \dots q_{m_s} |-\beta+n, -\beta+n\rangle \qquad n_i, m_j < 0$$

The direct sum

$$\Pi_{-\beta} := \oplus_{n\in\mathbb{Z}} \Pi_{-\beta+n,-\beta+n}$$

has the structure of a $\Pi_0 \otimes_{\mathbb{C}} \mathbb{C}[\beta]$-module with the same formulas of the non quantum case.

Analogously we may define the π_0^β-module $\pi_{2\beta}^\beta$

Consider the vertex operator

$$V_{-\beta,2\beta}(z) := \widetilde{a}^{-\beta}(z)V_{2\beta}(z) : \Pi_0 \otimes_{\mathbb{C}} \pi_0^\beta \to \Pi_{-\beta} \otimes_{\mathbb{C}[\beta]} \pi_{2\beta}^\beta$$

where

$$\widetilde{a}^{-\beta}(z) = T_{-\beta,-\beta} \exp\Big(-\sum_{n<0} \frac{\beta p_n + \beta q_n}{n} z^{-n} \Big) \exp\Big(-\sum_{n>0} \frac{\beta p_n + \beta q_n}{n} z^{-n} \Big)$$

$$V_{2\beta}(z) = T_{2\beta} \exp\Big(-\sum_{n<0} \frac{b_n}{n} z^{-n} \Big) \exp\Big(-\sum_{n>0} \frac{b_n}{n} z^{-n} \Big)$$

We have the following proposition

Proposition 7.4.2. *The operator*

$$\overline{S}_\beta := \int V_{-\beta,2\beta}(w)dw$$

intertwines with the action of $\widehat{\mathfrak{sl}}_{2\beta}$ on both modules.

Proof. By corollary 7.3.1 the specialization of $\overline{S}_\beta$ at $\beta = k+2$ for $k \neq -2$ intertwines with the associated action of $\widehat{\mathfrak{sl}}_{2k}$ since it turns out to be exactly the operator $\overline{S}_k$. By lemma 7.4.1 then we see that the same is true for $\overline{S}_\beta$. □

Consider now the $\mathbb{C}$ vector subspaces

$$(\Pi_0 \otimes \pi_0^\beta)_{res} \subset \Pi_0 \otimes \pi_0^\beta \qquad (\Pi_{-\beta} \otimes \pi_{2\beta}^\beta)_{res} \subset \Pi_{-\beta} \otimes \pi_\beta^\beta$$

defined to be the $\mathbb{C}$-span of the monomials

$$p_{i_1} \dots p_{i_s} q_{j_1} \dots p_{j_r} b_{k_1} \dots b_{k_l} |n,n\rangle \otimes |0\rangle \qquad p_{i_1} \dots p_{i_s} q_{j_1} \dots p_{j_r} b_{k_1} \dots b_{k_l} |-\beta+m,-\beta+m\rangle \otimes |2\beta\rangle$$

respectively. Identify them both with the $\mathbb{C}$-vector space $\Pi_0 \otimes \pi_0$. Note in addition that the tensor product by $\mathbb{C}[\beta]$ on both spaces induce isomorphisms

$$(\Pi_0 \otimes \pi_0^\beta)_{res} \otimes \mathbb{C}[\beta] = \Pi_0 \otimes \pi_0^\beta \qquad (\Pi_{-\beta} \otimes \pi_{2\beta}^\beta)_{res} \otimes \mathbb{C}[\beta] = \Pi_{-\beta} \otimes \pi_\beta^\beta$$

Under these identifications we may expand an element $v \in \Pi_0 \otimes \pi_0^\beta$ (or in $\Pi_{-\beta} \otimes \pi_{2\beta}^\beta$) in powers of β

$$v = v_0 + \beta v_1 + \beta^2 v_2 + \dots$$

with $v_i \in \Pi_0 \otimes \pi_0$. Note that this sum is finite. Analogously given a $\mathbb{C}[\beta]$ linear homomorphism $f : \Pi_0 \otimes \pi_0^\beta \to \Pi_{-\beta} \otimes \pi_{2\beta}^\beta$ we may express it in powers of β

$$f = f_0 + \beta f_1 + \beta^2 f_2 + \dots$$

with $f_i \in \mathrm{End}(\Pi_0 \otimes \pi_0)$. Note that this sum is not necessarily finite but it becomes finite once we apply to it any vector $v \in \Pi_0 \otimes \pi_0$.

Lemma 7.4.2. *Given any* $A \in (\Pi_0 \otimes \pi_0^\beta)_{res}$ *and any* $X \in L\mathfrak{g} \subset \hat{\mathfrak{g}}_\beta$ *then*

$$X \cdot_\beta A = X \cdot_{k_c} A + \beta(\dots)$$

where $X \cdot_\beta A$ *denotes the action of* X *as an element of* $\hat{\mathfrak{g}}_\beta$ *while* $X \cdot_{k_c} A$ *denotes the action of* X *as an element of* $\hat{\mathfrak{g}}_{k_c}$ *on* $A \in \Pi_0 \otimes \pi_0$ *through the identification* $(\Pi_{-\beta} \otimes \pi_{2\beta}^\beta)_{res} = \Pi_0 \otimes \pi_0$.

Proof. This follows from the fact that the specialization of $\Pi_{-\beta} \otimes \pi_{2\beta}^\beta$ at $\beta = 0$ is equal to $\Pi_0 \otimes \pi_0$ as a $\Pi_0 \otimes \pi_0$ module. And from the remark that $\Pi_0 \otimes \pi_0 = (\Pi_0 \otimes \pi_0^\beta)_{res} \to \Pi_0 \otimes \pi_0^\beta/(\beta)$ is an isomorphism of vertex algebras. □

Consider now the expansion of $\overline{S}_\beta$ in powers of β.

$$\overline{S}_\beta = \sum_{n \geq 0} \overline{S}_\beta^n \beta^n$$

Proposition 7.4.3. *The first non zero factor of* $\overline{S}_\beta$ *in the above expansion, considered as an element in* $\mathrm{End}(\Pi_0 \otimes \pi_0)$ *intertwines with the action of* $\hat{\mathfrak{sl}}_{2k_c}$. *We call it* $\overline{S}$, *the* ***screening operator*** *at the critical level.*

Proof. Write

$$\overline{S}_\beta = \beta^n(\overline{S} + \beta(\dots))$$

and let $A \in \Pi_0 \otimes \pi_0$ any vector. Consider any element $X \in L\mathfrak{g}$ as in lemma 7.4.2. We have by proposition 7.4.2

$$X \cdot_\beta \overline{S}_\beta(A) = \overline{S}_\beta(X \cdot_\beta A)$$

expanding both expressions in powers of β, using lemma 7.4.2. the fact that both homomorphisms are $\mathbb{C}[\beta]$ linear and finally comparing the lowest degree terms we obtain

$$X \cdot_{k_c} \overline{S}(A) = \overline{S}(X \cdot_{k_c} A)$$

□

7.4.3 Computation of $\overline{S}$

We will expand separately the operators

$$\widetilde{a}^{-\beta}(z) \qquad \text{and} \qquad V_{2\beta}(z)$$

in terms of the identifications $\Pi_\beta = \Pi_0 \otimes \mathbb{C}[\beta]$ and $\pi_0^\beta = \pi_0 \otimes \mathbb{C}[\beta]$ as $\mathbb{C}[\beta]$-modules.

Remark that on $\Pi_\otimes \mathbb{C}[\beta]$ the operators p_n, q_n acts only on the first factor. On the other hand considering $\pi_0 \otimes \mathbb{C}[\beta] = \pi_0^\beta$, it is quite clear that the operators b_n acts as follow:

$$b_n = b_n \text{ if } n < 0 \qquad b_n = 2n\beta \frac{\partial}{\partial b_{-n}} \text{ if } n > 0$$

We start with $V_{2\beta}(z)$. Write $V_{2\beta}(z) = \sum_{n\in\mathbb{Z}} V_{2\beta}[n]z^{-n}$. Define in addition

$$\sum_{n\leq 0} \overline{V}[n]z^{-n} := \exp\biggl(-\sum_{n<0} \frac{b_n}{n} z^{-n}\biggr)$$

By the remark above we obtain the following expansion

$$\exp\biggl(-\sum_{n>0} \frac{b_n}{n} z^{-n}\biggr) = \exp\biggl(-\sum_{n>0} 2\beta \frac{\partial}{\partial b_{-n}} z^{-n}\biggr) = 1 + -2\beta\biggl(\sum_{n>0} \frac{\partial}{\partial b_{-n}} z^{-n}\biggr) + \beta^2(\dots)$$

So we have

$$V_{2\beta}(z) = \sum_{n\leq 0} \overline{V}[n]z^{-n} - 2\beta\biggl(\sum_{n\leq 0} \overline{V}[n]z^{-n}\biggr)\biggl(\sum_{m>0} \frac{\partial}{\partial b_{-m}} z^{-m}\biggr)$$

and we find that if we write $V_{2\beta}(z) = \sum_{n\in\mathbb{Z}} \overline{V}_{2\beta}[n]z^{-n}$

$$\begin{aligned} \overline{V}_{2\beta}[n] &= \overline{V}[n] + \beta(\dots) \qquad \text{if } n \leq 0 \\ \overline{V}_{2\beta}[n] &= \beta\overline{V}[n] + \beta^2(\dots) \text{ if } n < 0 \end{aligned}$$

where for $n < 0$

$$\overline{V}[n] := -2\sum_{m\leq 0} \overline{V}[m] \frac{\partial}{\partial b_{m-n}}$$

On the other hand we have

$$\widetilde{a}^{-\beta}(z) = \biggl(1 + \beta\sum_{n<0} \frac{p_n + q_n}{n} z^{-n} + \beta^2(\dots)\biggr)\biggl(1 + \beta\sum_{n>0} \frac{p_n + q_n}{n} z^{-n} + \beta^2(\dots)\biggr)$$

from which, denoting $\widetilde{a}^{-\beta}(z) = \sum_{n\in\mathbb{Z}} \widetilde{a}^{-\beta}[n]z^{-n}$

$$\begin{aligned} \widetilde{a}^{-\beta}[n] &= 1 + \beta(\dots) \text{if } n = 0 \\ \widetilde{a}^{-\beta}[n] &= \beta\biggl(\frac{p_n + q_n}{n}\biggr) + \beta^2(\dots) \end{aligned}$$

It immediately follows the following formula for the residue $\overline{S}_{2\beta}(z)$

$$\overline{S}_\beta = \beta\biggl(\overline{V}[1] + \sum_{n>0} \frac{1}{n}\overline{V}[-n+1](p_n + q_n)\biggr) + \beta^2(\dots)$$

we proved the following proposition

Proposition 7.4.4. *Let $\overline{V}[n]$ be the operators above, then*

$$\overline{S} = \overline{V}[1] + \sum_{n>0} \frac{1}{n}\overline{V}[-n+1](p_n + q_n)$$

In particular it maps $\pi_0 \to \pi_0$ and on this space it holds the following formula

$$\overline{V} := \overline{S}_{|\pi_0} = \overline{V}[1] = -2 \sum_{n \le 0} \overline{V}[n] \frac{\partial}{\partial b_{n-1}}$$

Notice that $\overline{S}$ kills the vacuum vector $|0\rangle \in \Pi_0 \otimes \pi_0$.

7.5 Screening operators for arbitrary $\mathfrak{g}$

We want to extend the previous result to an arbitrary simple Lie algebra $\mathfrak{g}$. In order to do so we will use the functor of semi infinite parabolic induction introduced in Theorem 6.5.1. In order to do so we consider the parabolic subalgebras

$$\mathfrak{p}^{(i)} \subset \mathfrak{g} \qquad \mathfrak{p}^{(i)} := \mathfrak{b}_- \oplus \mathbb{C} e_i$$

in the decomposition $\mathfrak{p}^{(i)} = \mathfrak{m} \oplus \mathfrak{r}$ we may take

$$\mathfrak{m} = \mathfrak{sl}_2^{(i)} \oplus (h_i)^\perp$$

by theorem 5.7.1 there is an homomorphism of vertex algebras

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes V_{\kappa_c}(\mathfrak{sl}_2) \otimes \pi_0((h_i)^\perp)$$

The following proposition follows directly from the constructions

Proposition 7.5.1. *Consider the isomorphisms $M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes M \simeq M_{\mathfrak{g}}$ and $\pi_0(\mathbb{C}h_i) \otimes \pi_0((h_i)^\perp) \simeq \pi_0(\mathfrak{h})$ then the composition*

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes V_{\kappa_c}(\mathfrak{sl}_2) \otimes \pi_0((h_i)^\perp) \to M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes M \otimes \pi_0(h_i) \otimes \pi_0((h_i)^\perp) \simeq M_{\mathfrak{g}} \otimes \pi_0(\mathfrak{h})$$

is exactly the free field realization of $V_{\kappa_c}(\mathfrak{g})$ of Theorem 5.5.1.

To apply the semi-infinite induction functor to this case we need to extend the definition of $\overline{S}$ in order to make it an intertwining operator of a $\hat{\mathfrak{sl}}_{2\,\kappa_c} \oplus \widehat{(h_i)^\perp}$ action.

It suffices to consider the operator

$$\overline{S} \otimes \mathrm{id} : M \otimes \pi_0(h_i) \otimes \pi_0((h_i)^\perp) \to \Pi_0 \otimes \pi_0(h_i) \otimes \pi_0((h_i)^\perp)$$

It intertwines with the action of $\hat{\mathfrak{sl}}_{2\,\kappa_c} \oplus \widehat{(h_i)^\perp}$

Definition 7.5.1. For an arbitrary $\mathfrak{g}$ we define the screening operators $\overline{S}_i$ as the operators obtained through the functor of semi-infinite parabolic for the subalgebra $\mathfrak{p}^{(i)} \subset \mathfrak{g}$. It is clear from the construction that $\overline{S}_i(\pi_0(\mathfrak{h})) \subset \pi_0(\mathfrak{h})$ we call

$$\overline{V}_i := (\overline{S}_i)_{|\pi_0(\mathfrak{h})}$$

Proposition 7.5.2. *The image of the embedding*

$$V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0(\mathfrak{h})$$

is contained in the intersection

$$\bigcap_{i=1,\dots,l} \ker \overline{S}_i$$

Proof. The operators $\overline{S}_i$ commute with the action of $\hat{\mathfrak{g}}_{\kappa_c}$ and kill the vacuum vector. □

Consider now the isomorphism

$$\pi_0(\mathfrak{h}) \to \pi_0(h_i) \otimes \pi_0((h_i)^\perp) \qquad b_{i,} \mapsto b_{i,n} \qquad b_{j,n} \mapsto \frac{1}{2}a_{ji}b_{i,n} + (b_{j,n} - \frac{1}{2}a_{ji}b_{i,n})$$

Proposition 7.5.3. *The operator* $\overline{V}_i : \pi_0 \to \pi_0$ *is characterized by the following formula:*

$$\overline{V}_i = -\sum_{m\geq 0} \overline{V}[m]\boldsymbol{D}_{b_{i,m-1}}$$

where $\boldsymbol{D}_{b_{i,m-1}}$ *is a derivation of* π_0 *such that*

$$\boldsymbol{D}_{b_{i,m}} \cdot b_{j,m} = a_{ji}\delta_{n,m}$$

Proof. Recall that the operator $\overline{V}_i$ is $\overline{V} \otimes \mathrm{id}$ and that

$$\overline{V} = -2\sum_{m\geq 0} \overline{V}[m]\frac{\partial}{\partial b_{i,m-1}}$$

Using the isomorphism above between $\pi_0(\mathfrak{h}) \simeq \pi_0(h_i) \otimes \pi_0((h_i)^\perp)$ we find

$$\frac{\partial}{\partial b_{i,n}} \cdot b_{j,m} = \frac{\partial}{\partial b_{i,n}} \cdot (b_{j,m} - \frac{1}{2}a_{ji}b_{i,m}) + \frac{1}{2}a_{ji}\frac{\partial}{\partial b_{i,n}} \cdot b_{i,m} = 0 + \frac{1}{2}a_{ji}\delta_{n,m}$$

where we used the fact that for $x \in \pi_0((h_i)^\perp)$ the derivative with respect to b_i is by definition 0. □

Proposition 7.5.4. *The center of the vertex algebra* $\zeta(\mathfrak{g})$ *is contained in the intersection*

$$\zeta(\mathfrak{g}) \subset \bigcap_{i=1,\ldots,l} \ker \overline{V}_i$$

Proof. We know that $\zeta(\mathfrak{g}) \subset \pi_0$ and that $V_{\kappa_c}(\mathfrak{g}) \subset \cap \ker \overline{S}_i$. It immediately follows that

$$\zeta(\mathfrak{g}) \subset \bigcap_{i=1,\ldots,l} \ker(\overline{S}_i)_{|\pi_0} = \bigcap_{i=1,\ldots,l} \ker \overline{V}_i$$

□

Chapter 8

Identification with the algebra of functions on the space of Opers

This section is devoted to the identification of the center of the vertex algebra $\zeta(\mathfrak{g}) \subset V_{k_c}(\mathfrak{g})$ with the algebra of functions on the space of Opers which is a classifying space for certain connections on the trivial G-bundle on D.

We will start by studying principal bundles and connections on them, we proceed by defining the space of Opers and Miura Opers and finally link the algebras of functions on these spaces to the center $\zeta(\mathfrak{g})$.

8.1 Principal Bundles

We first have to define what a principal bundle is and what is a connection on it. Some of the definitions we will be giving are geometric but sometimes the Tannakian formalism, and therefore a more functorial point of view, will come in handy.

From now on X will denote a scheme over a $\mathbb{C}$ algebra R, G an algebraic group over $\mathbb{C}$, G_R will denote its extension of scalars to R. All fibered products are to be understood over the most natural space.

Definition 8.1.1. A *principal* G*-bundle* over X is a scheme P with a right action of G: $\mu : P \times G \to P$ and a projection $p : P \to X$ which is G invariant. We also require this bundle to be locally trivial:

There exist a covering U_i of X for the Zariski topology for which $P_{|U_i} \simeq U_i \times G$ in a G equivariant way.

A principal bundle P that is isomorphic to $X \times G$ (always in a G equivariant way) will be called trivial.

To shorten our notation we will call a principal G bundle simply a G bundle. It is easy to check that given a map of schemes $f : Y \to X$ the fibered product $f^*P = Y \times_X P$ is naturally a G bundle.

Remark 8.1.1. We state some simple remarks:

- The natural map $P \times G \to P \times_X P$ which sends $(p, g) \to (p, pg)$ is an isomorphism. This may be checked locally on X, and it is quite easy to see for the trivial bundle;

- A map of G bundles is a G-equivariant morphism $\varphi : P_1 \to P_2$ such that $p_2 \circ \varphi = p_1$. Such a map must be an isomorphism since the fibers are single G-orbits. A map satisfying these properties will be simply called an isomorphism;

- A principal bundle P over X is trivial if and only if the structural morphism $P \to X$ admits a section. The set of such sections will be denoted by $P_X(X)$, it may be checked that it is a $G(X)$ torsor (i.e. it has a natural $G(X)$ action which is simply transitive);

- The set of automorphisms of the trivial G bundle $X \times G$ is isomorphic to the set $G(X) = \mathrm{Hom}(X, G)$. Indeed consider a G equivariant isomorphism that commutes with the projections $\Phi : X \times G \to X \times G$, observe that since Φ is G equivariant and it commutes with the projections it must of the form $(x, g) \mapsto (x, \varphi(x)g)$ for some $\varphi : X \to G$.

- Given a G-scheme Y we can consider the product $P \times Y$, it carries a natural right G action, defined by $(p, y) \cdot g := (pg, g^{-1}y)$ and since P is locally trivial the quotient $P(Y) := (P \times Y)/G$ exists and it is naturally a Y-bundle over X. By a slight abuse of notation we will denote this quotient with the same symbol used for the fibered product: $P \times_G Y$. We will also use this notation when we are in other situations when the subscript is a group we will mean that we are considering the quotient space by the prescribed group action.

Definition 8.1.2. Given H a subgroup of G, and a principal H bundle P over X the scheme $P(G)$ admits a natural right G action that makes it into a principal G bundle. Given a G bundle P over X a H-**reduction** of P is a principal H bundle P_H with an isomorphism $P_H(G) \simeq P$

Note that the map $P_H \to P_H \times G \to P_H(G) \simeq P$ which sends $p \mapsto (p, 1) \mapsto [p, 1]$ is clearly H equivariant since $[ph, 1] = [p, h] = [p, 1]h$. We deduce that up to isomorphism to give an H-reduction is equivalent to give an H-bundle P_H and an H-equivariant map $P_H \to P$. Such a map is easily checked to be an embedding. Thanks to this remark we will always speak of an H-reduction as an H-invariant embedded sub-bundle $P_H \subset P$.

We now focus on the trivial case. Consider the trivial G-bundle $X \times G$, an H-trivial reduction is therefore an H-equivariant embedding $\Phi : X \times H \to X \times G$. Consider the map $\varphi : X \to G$ defined by composing the embedding $p \mapsto (p, 1) \in X \times H$ with the projection $X \times G \to G$. This map determines the embedding since, by H-invariancy on R-points, Φ must be of the form

$$(p, h) \mapsto (p, \varphi(p)h) \qquad \text{for } p \in X(R), h \in H(R) \text{ and } \varphi(p) \in G(R)$$

Consider now another H-equivariant map $\Phi' : X \times H \to X \times G$ that defines the same reduction, or equivalently such that Φ and Φ' have the same image. Consider the corresponding morphisms $\varphi, \varphi' : X \to G$ defined as above. The fact that Φ and Φ' have the same image translates to the following condition on R-points:

$$\{\varphi(p)h : h \in H(R)\} = \{\varphi'(p)h : h \in H(R)\} \qquad \forall\, p \in X(R)$$

But this is equivalent to ask that $\varphi(p)\varphi'(p)^{-1} \in H(R)\, \forall\, p \in X(R)$. Or equivalently that there exists a map $\psi : X \to H$ such that $\varphi = \varphi' \cdot \psi$.

We just proved the following:

Proposition 8.1.1. *The set of trivial* H*-reductions of the trivial* G*-bundle over* X*,* $X \times G$ *is in natural correspondence with*

$$G(X)/H(X)$$

More generally, given a trivial G*-bundle* P *over* X *the set of* H*-reductions of* P *is in natural correspondence with:*

$$P_X(X) \times_{G(X)} G(X)/H(X)$$

Where $P_X(X)$ *is the set of sections of the projection map* $P \to X$*. It is not empty since we are assuming that* P *is trivial.*

Proof. The first statement follows from the preceding discussion. While the second statement is a natural generalization taking into account that the set of trivializations of P is equal to $P_X(X)$. $\square$

8.2 Connections on vector bundles

We now turn our attention to connections. We will define what a connection on a vector bundle is paying specific attention to the trivial case. In the next section the Tannakian formalism will allow us to extend this definition to connections on a principal G-bundle.

By a vector bundle over an R-scheme X we mean an R-scheme V with a projection $\pi : V \to X$ which is locally (in the Zariski topology) isomorphic to $X \times_R \mathbb{A}^n_R$ and such that the transition functions are R linear. The sheaf of sections of V is naturally a locally free sheaf of $\mathcal{O}_X$ modules of equal rank n. Every vector bundle is uniquely determined by its sheaf of sections, therefore we will make no difference between them and call them by the same symbols, each interpretation will be clear from the context.

Definition 8.2.1. Let X be an R-scheme. An R-**connection** on a vector bundle $V \to X$, is a morphism between the $\mathcal{O}_X$-modules:

$$\nabla : V \to V \otimes_{\mathcal{O}_X} \Omega^1_{X/R}$$

which is R-linear and satisfy the following Leibniz rule

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla(\sigma)$$

for any function $f \in \mathcal{O}_X$ and any section $\sigma \in V$.

Remark 8.2.1. It is easy to check from the definition that the difference between two connections is a homomorphism of $\mathcal{O}_X$-modules, in addition given an homomorphism of $\mathcal{O}_X$-modules $\sigma : V \to V \otimes \Omega^1_{X/R}$ and a connection ∇ the morphism $\nabla + \sigma : V \to V \otimes \Omega^1_{X/R}$ is still a connection.

We deduce from this that the space of connections, if not empty, is a torsor over $\mathcal{H}om_{\mathcal{O}_X}(V, V \otimes \Omega^1_{X/R})$.

Given a finite dimensional vector space V over $\mathbb{C}$ we attach to it a canonical functor $R \mapsto V(R) = V \otimes_{\mathbb{C}} R$ this is easily seen to be representable by $\mathbb{A}^{\dim V}$ and therefore is an affine scheme. We denote by V_a this scheme. V_a may be also defined over an arbitrary $\mathbb{C}$-algebra R by extension of scalars. We call this $V_{R,a}$, its points on a given R-algebra R' are just $V(R') = V \otimes_{\mathbb{C}} R' = (V \otimes_{\mathbb{C}} R) \otimes_R R'$.

Given a vector space and an R-scheme X we can define the trivial vector bundle with fiber V as $X \times_R V_{a,R}$. Its sheaf of sections is canonically isomorphic to $V \otimes_{\mathbb{C}} \mathcal{O}_X$.

Connections on trivial bundles are particularly easy to describe. Indeed consider the trivial bundle $X \times_R V_{R,a}$ for a given finite dimensional vector space. Consider the canonical connection

$$d : V_R \otimes_R \mathcal{O}_X \to (V_R \otimes_R \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^1_{X/R} = V_R \otimes_R \Omega^1_{X/R}$$

Induced by the canonical differential $d : \mathcal{O}_X \to \Omega^1_{X/R}$ and extended by linearity.

We know by Remark 8.2.1 that any connection is of the form $d + A$ where $A \in \mathcal{H}om_{\mathcal{O}_X}(V_R \otimes_R \mathcal{O}_X, V_R \otimes_R \Omega^1_{X/R}$ but this last space is isomorphic to $\mathcal{E}nd_R(V_R) \otimes_R \Omega^1_{X/R}(X)$. We proved

Proposition 8.2.1. *Every connection on the trivial bundle $X \times_R V_{R,a}$ is of the form*

$$d + A \qquad \textit{with} \quad A \in \mathcal{E}nd_R(V_R) \otimes_R \Omega^1_{X/R}(X)$$

The category $\mathbf{Bun}_X$ of vector bundles over X is naturally an additive tensor category and connections satisfy some nice properties relatively to the tensor structure. In particular the following hold:

Proposition 8.2.2. *Let (V, ∇_V), (W, ∇_W) two vector bundles with connection over an R-scheme X.*

- *The vector bundle $V \otimes W$ has a natural connection namely $\nabla_V \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_W$ which is defined as follows:*

  $$\nabla_V \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_W : V \otimes W \to V \otimes W \otimes \Omega^1_{X/R} \qquad \sigma_1 \otimes \sigma_2 \mapsto \nabla_V(\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes \nabla_W(\sigma_2)$$

  It is easily checked to be a connection

- *The vector bundle $V^\vee$ carries a natural connection $\nabla_V^\vee$ defined by the following formula:*

  $$d\langle \varphi, \sigma \rangle = \langle \nabla_V^\vee(\varphi), \sigma \rangle + \langle \varphi, \nabla(\sigma) \rangle$$

  for any $\varphi \in V^\vee$, $\sigma \in V$

We now study how connections on trivial bundles change under different choices of trivialization. In particular choose a vector bundle isomorphism $g : X \times V_{R,a} \to X \times V_{R,a}$, this is equivalent to give a map $g : X \to GL_R(V_R)$ or equivalently $g \in GL_R(V_R)(X)$.

Proposition 8.2.3. *Consider a connection $\nabla = d + A$ on the trivial vector bundle $X \times_R V_{R,a}$ and an automorphism $g \in GL_R(V_R)(X)$, then under this automorphism the connection is read as*

$$g \cdot \nabla = d + gAg^{-1} - dg \cdot g^{-1}$$

Let's explain what does this mean. Thanks to proposition 8.2.1 we may think of A as a matrix with coefficients in $\Omega^1_{X/R}(X)$ while g may be thought as an invertible matrix with coefficients in $\mathcal{O}_X(X)$.

Then dg makes sense as a matrix with coefficients in $\Omega^1_{X/R}(X)$ and it makes sense to multiply (both from left and right) matrices with 1-forms coefficients by matrices with functions coefficients so gAg^{-1} and $dg \cdot g^{-1}$ both make sense;.

Proof. Fix a basis of V let's call it $e_1, \dots, e_n$. The vectors e_i also define sections of the associated vector bundle $e_i = e_i \otimes 1$. The matrix A is easily computed evaluating ∇ on these sections: if $\nabla(e_i) = \sum_j e_j \omega_{ji}$ for some 1-forms $\omega_{ij} \in \Omega^1_{X/R}(X)$ then we have $A_{ij} = \omega_{ij}$ indeed

$$\nabla(\sum_i f_i e_i) = \sum_i e_i df_i + \sum_i f_i \nabla(e_i) = \sum_i e_i df_i + \sum_{ij} e_j \omega_{ji} f_i = d(\sum_i f_i e_i) + A \cdot (\sum_i e_i f_i)$$

To compute the matrix associated to the basis $s_i = g e_i$ we must evaluate ∇ on these sections and express it in the same basis.

The rest of the proof is a straightforward computation. □

8.3 Tannakian Formalism

We now give another description of principal G bundles over a scheme X. Loosely speaking we will state a version of the Tannakian formalism which describes P in term of certain tensor functors.

Given a principal G-bundle P and $V \in \mathbf{Rep}_{\mathbb{C}}(G)$ a G representation we may construct as above a bundle $P(V)$, this turns out to be a vector bundle over X and given two representations V and W there are natural isomorphism $P(V \oplus W) \simeq P(V) \oplus P(W)$, $P(V \otimes W) = P(V) \otimes P(W)$.

Call $\mathcal{P} : \mathbf{Rep}_{\mathbb{C}}(G) \to \mathbf{Bun}_X$ the functor from the category $\mathbf{Rep}_{\mathbb{C}}(G)$ of G representations of finite dimension to the category $\mathbf{Bun}_X$ of vector bundles over X that sends $V \mapsto P(V)$.

For the definition of rigid tensor category and the classical Tannakian formalism refer to [DM82]. We couldn't find a precise reference which shows the following theorem, but we cite [FBZ04][Section 1.2.4] where it is stated as a fact.

Proposition 8.3.1. *The functor $\mathcal{P}$ is an additive exact tensor functor between rigid tensor categories.*

Theorem 8.3.1. *The association $P \mapsto \mathcal{P}$ is fully faithful.*

We will therefore denote by the letter P the functor associated with the G-bundle P, and thanks to the above theorem we will make no difference between the principal bundle P and its associated functor. Thanks to this formalism we are able to transpose the definition of a connection from the "linear" setting of vector bundles to the "group" setting.

Definition 8.3.1. A **connection** on a principal bundle $P : \mathbf{Rep}_{\mathbb{C}}(G) \to \mathbf{Bun}_X$ is a family of connections ∇_V parametrized by the representations of G on the vector bundle $P(V)$ such that $\nabla_{V \oplus W} = \nabla_V \oplus \nabla_W$ and $\nabla_{V \otimes W} = \nabla_V \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_W$ in the sense of proposition 8.2.2 using the provided identification $P(V \otimes W) = P(V) \otimes P(W)$

We focus one again on the trivial setting. Let $P_{can} : \mathbf{Rep}_{\mathbb{C}}(G) \to \mathbf{Bun}_X$ the canonical trivial principal bundle $V \mapsto X \times_{\mathbb{C}} V_a$. Then giving a connection on P_{can} is equivalent, by proposition 8.2.1 a family of elements $\nabla_V \in \mathrm{End}_{\mathbb{C}}(V) \otimes \Omega^1_{X/\mathbb{C}}(X)$ satisfying the property $\nabla_{V \otimes W} = \nabla_V \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_W$ given a linear functional $\xi \in \mathrm{Hom}_{\mathbb{C}[X]}(\Omega^1_{X/\mathbb{C}}, \mathbb{C}[X])$ this condition express as follows. We call $\nabla_{V,\xi}$ the endomorphism of V obtained by contraction with ξ. Then we have

$$\nabla_{V \otimes W, \xi} = \nabla_{V,\xi} \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_{W,\xi}$$

Where by $\nabla_{V,\xi} \otimes \mathbf{1}$ we mean the endomorphism of $V \otimes W$ defined by tensoring $\nabla_{V,\xi}$ with the identity. Using the Tannakian formalism for Lie algebras (see [DM82]), we obtain that the datum $V \mapsto \nabla_{V,\xi}$ defines a unique element $\nabla_\xi \in \mathfrak{g} \otimes \mathbb{C}[X]$, this association is clearly $\mathbb{C}[X]$-linear in ξ. If in addition the pairing

$$\Omega^1_{X/\mathbb{C}} \times \mathrm{Hom}_{\mathbb{C}[X]}(\Omega^1_{X/\mathbb{C}}, \mathbb{C}[X]) \to \mathbb{C}[X]$$

is perfect we may glue all these elements to a unique operator $\nabla \in \mathfrak{g} \otimes_{\mathbb{C}} \Omega^1_{X/\mathbb{C}}(X)$. We just proved the following proposition.

Proposition 8.3.2. *If $(\Omega^1_{X/\mathbb{C}})^{\vee\vee} = \Omega^1_{X/\mathbb{C}}$ then to give a connection on the principal trivial bundle P_{can} is equivalent to give an element of*

$$\nabla \in \mathfrak{g} \otimes_{\mathbb{C}} \Omega^1_{X/\mathbb{C}}(X)$$

Along with this information we may restate proposition 8.2.3 in the group setting, analysing how changing the trivialization for our principal bundle changes the connection;

Proposition 8.3.3. *Suppose* X *satisfies the condition of the above proposition and let* $P = X \times G$ *the trivial* G*-bundle. And let* $g \in G(X)$ *be an automorphism of* P*. Consider a given connection on* P*:*

$$\nabla = d + A \qquad \textit{with } A \in \mathfrak{g} \otimes \Omega$$

Then under the trivialization induced by g *the connection takes the form*

$$g \cdot \nabla = d + Ad_g(A) - dg \cdot g^{-1}$$

We will call this action the ***gauge*** *action of* $G(X)$ *on the space of connections.*

8.4 The case of the formal disc

We now define what a connection on a vector bundle is in the case of the formal disc. The only subtle definition is that is more convenient to impose that the space of 1-form must be $\Omega^{1,cont}_{D_R/R}$.

Definition 8.4.1. Let $X = D_R$ (we do not choose a coordinate here) and let V be a vector bundle on X. An R-connection on V is a morphism of $\mathcal{O}_X$ modules:

$$\nabla : V \to V \otimes \Omega^{1,cont}_{D_R/R}$$

which is R-linear and satisfy the following Leibniz rule:

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla(\sigma)$$

for any function $f \in \mathcal{O}_X$ and any section $\sigma \in V$.

The other definitions are identical to the one we defined above. So we have principal G-bundles on D_R, and connections on them defined as family of connections on the associated vector bundles.

We consider R-groups which are actually defined over $\mathbb{C}$, since the space $\Omega^{1,cont}_{D_R/R}$ is free of rank 1 its bidual is canonically isomorphic to it hence as in proposition 8.3.2 the space of connections on the trivial principal G-bundle is in correspondence with the elements in

$$\mathfrak{g} \otimes \Omega^{1,cont}_{D_R/R} = \mathfrak{g} \otimes R[[t]]dt$$

8.4.1 Action of coordinate changes

We just saw that we may express a connection on a trivial G-principal bundle may expressed as an element of $\mathfrak{g} \otimes R[[t]]dt$.This expression is dependent on the choice of the coordinate t, we now wonder how the same connection is expressed in terms of another coordinate s such that $t = \rho(s)$.

We first consider the easier case of a trivial vector bundle. Let $V = \mathcal{O}^n_D$. The matrix element defining the connection is expressed with coefficients in $\Omega^{1,cont}_{R[[t]]/R}$ and it corresponds to the evaluation of the connection on the constant sections e_i. The latter sections of our vector bundle do not change under change of coordinates since they are constant. We only have to express one forms in the new coordinate.

Remark 8.4.1. Let $\omega = f(t)dt \in \Omega^{1,cont}_{R[[t]]/R}$ then under a change of coordinate $t = \rho(s)$ (or equivalently under the automorphism induced by ρ) ω is read as

$$\omega = f(\rho(s))\rho'(s)ds$$

Note that $\rho'(s)$ is automatically invertible (as an element of $R[[s]]$) since its leading coefficient is invertible in R.

This discussion extends naturally to the context of G-trivial principal bundles. Applying the automorphism $\rho(t)$ rewrites for each vector bundle with a connection $d + A(t)dt$ a new connection $d + A(\rho(s))\rho'(s)ds$ and this formula defines an action on the space of G-connections.

Proposition 8.4.1. *Let* $P = D_R \times G$ *the trivial vector bundle over* D_R*. Then we have a natural* $\mathrm{Aut}\,\mathcal{O}(R)$ *action on the space of connections on* P *which is described by the following formula:*

$$\rho(t) \cdot \big(d + A(t)dt\big) = d + A(\rho(t))\rho'(t)dt$$

8.5 Opers

We restrict ourselves to the case of the formal disc. From now on G will be a reductive group defined over $\mathbb{C}$ of adjoint type, its Lie algebra will be denoted by $\mathfrak{g}$. We have a closed immersion $G \to GL(\mathfrak{g})$. We fix a torus and a Borel subgroup of G, $H \subset B \subset G$ which gives us a Cartan subalgebra and a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. This gives us a basis (α_i) for the root system and We also fix generators $f_i \in \mathfrak{g}_{-\alpha_i}$ and call $p_{-1} = \sum_i f_i$.

In what follows we are going to define the functor $\mathrm{Op}_G(D)$ as a functor of $\mathbb{C}$-algebras classifying certain connections.

We start with a definition concerning reductions of bundles.

Definition 8.5.1. Let H be a subgroup of G and let ∇ be a connection on the trivial principal bundle $X \times G$. Let $\varphi : X \to G$ a morphism defining an H-reduction of $X \times G$. We say that ∇ preserves this reduction if

$$\varphi^{-1} \cdot \nabla \in d + u \text{ with } u \in \mathfrak{h} \otimes \Omega^1_{X/R}$$

This is to say that we require that when we bring the couple $(X \times G, \varphi)$ to the trivial reduction $(X \times G, 1)$ the connection must be with coefficients in the Lie algebra of H. This is an intrinsic property (i.e. it does not depend on the choice of φ) since we can only change varphi by right multiplication with an element of $H(X)$ and the gauge action of the latter group preserves connections of the above form.

Definition 8.5.2. An R-**Oper** over D is a triple (P, ∇, P_B) where P and P_B are principal trivial G and B bundles on D_R respectively, $P_B \subset P$ is a B reduction of P and ∇ is an R-connection such that given any trivialization of $P \simeq D_R \times_{\mathbb{C}} G$ such that $P_B = D_R \times_{\mathbb{C}} B \subset D_R \times_{\mathbb{C}} G$ the connection takes the form

$$\nabla = d + (\sum_i f_i \otimes \psi_i + \nu(t))dt \qquad \text{with } \psi_i \in (R[[t]])^* \text{ and } \nu(t) \in \mathfrak{b} \otimes R[[t]]$$

An isomorphism of Opers over D_R, (P, ∇, P_B) and (P', ∇', P'_B) is an isomorphism $P \simeq P'$ which preserves the additional structures (i.e. sends ∇ to ∇' and P_B to P'_B).

We remark here that the definition is consistent, suppose we have an isomorphism $D_R \times G \to D_R \times G$ which preserves $D_R \times B$. Such an isomorphism is given by a map $\varphi : D_R \to G$ or equivalently an element of $G(R[[t]])$. Since we are asking that $D_R \times P_B$ is sent to $D_R \times P_B$ the identity in $B(R[[t]])$ must be sent to an element of $B(R[[t]])$ but this is the image of φ so actually we have $\varphi \in B(R[[t]])$ and it is easy to check that the gauge action of $B(R[[t]])$ on the above form is still a connection of the same form.

We are ready to define the space of Opers on the formal disc relative to a group G.

Definition 8.5.3. We define $\mathrm{Op}_G(D)$ as the functor of $\mathbb{C}$-algebras:

$$\mathrm{Op}_G(D)(R) := \{R - \text{Opers on } D\}/\text{isom}$$

to a morphism of $\mathbb{C}$-algebras $f : R \to R'$ we associate the map between the corresponding Opers sending a triple (P, ∇, P_B) defined over R to the triple $(P_{R'}, \nabla_{R'}, (P_B)_{R'})$ where the bundles are obtained through the base change $D_{R'} \to D_R$ and the connection is the pullback connection which may be defined as

$$\nabla = d + \nu(t) \in d + \mathfrak{g}(R[[t]]) \mapsto f^*\nabla = d + f_*\nu(t) \in d + \mathfrak{g}(R'[[]t]])$$

Theorem 8.5.1. *The functor* $\mathrm{Op}_G(D)$ *is representable by an affine scheme over* $\mathbb{C}$.

In order to prove the above theorem we will need a couple of lemmas. We state first an easy remark that follows from the discussion after definition 8.5.2.

Remark 8.5.1. The functor $\mathrm{Op}_G(D)$ is naturally isomorphic to the functor

$$R \mapsto \{\sum_i f_i \otimes \psi_i + \nu(t) : \ \psi_i \in R[[t]]^* \text{ and } \nu(t) \in \mathfrak{b}[[t]]\}/B(R[[t]])$$

Where $B(R[[t]])$ acts on the above space through the gauge transformations defined in Proposition 8.3.3. Indeed we have a functorial map from the functor above, associating to an isomorphism class of $\sum_i f_i \otimes \psi_i + \nu(t)$ the Oper $(D_R \times G, d + \sum_i f_i \otimes \psi_i + \nu(t), D_R \times B)$ and the discussion above shows that this map induces a bijection on the isomorphism classes.

Now recall the isomorphism $B = H \times N$, so that to an element $x \in B(R[[t]])$ we can associate a couple $(t, n) \in H(R[[t]]) \times_{\mathbb{C}} N(R[[t]])$. We first focus on the torus H, since G is of adjoint type the pairing between the coroot lattice $X^\vee$ and the root lattice X is perfect. Recall that the functors $R \mapsto H(R)$ and $R \mapsto X^\vee \otimes_{\mathbb{Z}} R^*$ are naturally isomorphic and call $\omega_i^\vee$ the elements in $X^\vee$ defined by $\omega_i^\vee(\alpha_j) = \delta_{ij}$.

It is easy to see that the action of $N(R[[t]])$ does not change the addendum $\sum_i f_i \otimes \psi_i$ since its action only increases the weight. So only $H(R[[t]])$ acts on that addendum and there is a unique element, namely $\sum_i \omega_i^\vee \otimes \psi_i^{-1}$, such that the gauge action of $H(R[[t]])$ puts the connection in the form

$$p_{-1} \otimes 1 + \nu(t) \text{ with } \nu(t) \in \mathfrak{b} \otimes R[[t]]$$

Lemma 8.5.1. *The action of* $N(R[[t]])$ *on the space of connections of the form*

$$d + (p_{-1} \otimes 1 + \nu(t))dt \textit{ with } \nu(t) \in \mathfrak{b} \otimes R[[t]]$$

is free. In particular given such a connection there exists unique elements $U \in \mathfrak{n}(R[[t]])$ *and* $\tilde{v}(t) \in V \otimes R[[t]]$ *such that*

$$d + (p_{-1} \otimes 1 + v(t))dt = \exp(U) \cdot (d + (p_{-1} \otimes 1 + \tilde{v}(t))dt)$$

where V *is the space defined in the preliminary section concerning exponents of a Lie algebra.*

Proof. See [Fre07][Lemma 4.2.2] □

Corollary 8.5.1. *The functor of Opers* $\mathrm{Op}_G(D)$ *is naturally isomorphic to the functor*

$$R \mapsto \left\{ d + (p_{-1} \otimes 1 + v(t))dt : v(t) \in V \otimes R[[t]] \right\}$$

and therefore is representable. We will denote this expression for an Oper connection on $D_R \times G$ *the 'canonical' form.*

Proof. The natural map

$$d + (p_{-1} \otimes 1 + v(t))dt \mapsto (D_R \times G, d + (p_{-1} \otimes 1 + v(t))dt, D_R \times B)$$

is an isomorphism by the above discussion. □

We will now investigate the action of the group $\mathrm{Aut}\mathcal{O}$ on the functor of Opers. To represent the space of Opers we chose a particular expression for the connection which is dependent of a choice of a uniformizing parameter for the disc D, in other words a coordinate. Changing coordinate will not change the intrinsic connection itself but it will affect, as we are going to see now, our preferred description of it.

Consider an Oper connection $d+(p_{-1}+v(t))dt$ applying the action discussed in 8.4.1 we obtain that

$$\rho(t) \cdot \big(d + (p_{-1} + v(t))dt\big) = d + (p_{-1}\rho'(t) + v(\rho(t))\rho'(t))dt$$

This is of course still an Oper connection (we only applied a change of coordinate) but it is not expressed anymore in our preferred system of coordinates. Applying the Gauge action of $H(R[[t]])$, in particular of $\sum_i \omega_i^\vee \rho'(t) = p^\vee \rho'(t)$ we obtain

$$d + p_{-1} + Ad_{p^\vee \rho'(t)}(v(\rho(t))\rho'(t)) - p^\vee \frac{\rho''(t)}{\rho'(t)}$$

And finally to bring it in the canonical form we can apply the gauge action of $N(R[[t]])$. The formulas for this action are rather complicated and may be found in [Fre07].

8.6 Miura Opers

We defined what an Oper is and now our goal is to describe the center of the vertex algebra $\zeta(V_k(\mathfrak{g})) \subset V_{k_c}(\mathfrak{g})$ as the algebra of functions on the space of Opers. In order to do so we are now introducing an auxiliary space, the space $\mathrm{MOp}_G(D)$ of Miura Opers on the formal disc (we are really interested in the subspace $\mathrm{MOp}_G(D)_{gen}$ of *generic* Miura Opers).

Recall that we embedded the center $\zeta(V_k(\mathfrak{g}))$ in a bigger commutative algebra π_0.We will give a geometric interpretation of π_0 as the algebra of functions on the space of generic **Miura Opers** on the disc. The space of generic Miura Opers on the disc turns out to be a N-torsor over the space of ordinary Opers, using this information we will be able to embed the algebra of functions on the space of Opers as the intersection of the kernels of certain operators on the algebra of functions on the space of generic Miura Opers. This description allows us to compute the character of the ring of functions $\mathbb{C}[\mathrm{Op}_{^LG}(D)]$ and having embedded both $\zeta(V_k(\mathfrak{g}))$ and $\mathbb{C}[\mathrm{Op}_{^LG}(D)]$ in the same space we will be able to show that they are indeed equal.

Definition 8.6.1. An R-**Miura Oper** on the formal disc for the group G is a quadruple $(P, \nabla, P_B, P_{B_-})$ where the triple (P, ∇, P_B) is an R-Oper (recall that the principal bundles are defined over D_R) and P_{B_-} is another B_--reduction for the lower Borel subgroup, which we require to be trivial and preserved by ∇.

An isomorphism of R-Miura Opers is an isomorphism between the principal G-bundles which preserves the rest of the structure.

We define now the space of Miura Opers as a functor like we did with the space of Opers. There will be a slight difference though: we will see below that the functor we define is not representable, but its sheafification in the fpqc topology is.

Definition 8.6.2. The space $\mathrm{MOp}_G(D)$ of Miura Opers on the formal disc for the group G is defined as the functor of R-algebras:

$$\mathrm{MOp}_G(D)(R) = \{R - \text{Miura Opers}\}/\text{isom}$$

So a Miura Oper is just an Oper with some additional data. We are going to analyze how a Miura Oper relates with the underlying Oper (i.e. the first three terms of the quadruple).

We first state a useful technical lemma.

Lemma 8.6.1. *Consider a group G and a connection on the trivial principal bundle $D_R \times X$ which we express as $\nabla = d + A(t)dt$ with $A(t) \in \mathfrak{g} \otimes R[[t]]$. Fix an element $g \in G(R)$.*

Then there exists a unique element $g(t) \in G(R[[t]])$ such that $g(0) = g$ and

$$g(t) \cdot \nabla = d$$

Proof. We must solve

$$\mathrm{Ad}_{g(t)}(A(t)) - g'(t)g^{-1}(t) = 0$$

where by $g'(t)$ we mean the derivative of $g(t)$ along t. Embedding G in the general linear group $G \to GL_n$ we may consider elements of G and of its Lie Algebra $\mathfrak{g}$ as matrices. The equation above reads as

$$g'(t) = g(t)A(t)$$

and being a linear differential equation it admits a unique solution with $g(0) = g$. This solution actually belongs to $G(R[[t]])$. This may be checked easily in matrix terms, indeed the conditions $g'(t)g^{-1}(t) \in \mathrm{Lie}(G)(R[[t]])$ and $g(0) \in G(R)$ imply that $g(t) \in G(R[[t]])$. □

We now are ready to analyze the structure of reductions preserved by a connections.

Proposition 8.6.1. *Let B be any subgroup of G and let P_B a trivial principal B- bundle over D_R, embedded in $D_R \times G$ and preserved by a connection ∇ on the latter G-bundle. Then $P_{B,0}$, determines P_B: in addition, given any B-reduction of $\operatorname{Spec} R \times G$ there exists a unique B reduction preserved by ∇ of $D_R \times G$ such that its restriction to the 0 point is the given reduction of $\operatorname{Spec} R \times G$.*

Proof. Thanks to the last lemma we may, up to isomorphism of $X \times G$, assume that the connection is the trivial connection. We prove the following equivalent statement: B-reductions preserved by the trivial connection on the trivial G-bundle $D_R \times G$ are in correspondence with B reductions of the pullback $\mathrm{Spec}\, R \times G$ at the 0 point $\mathrm{Spec}\, R \to D_R$. Consider a morphism $\varphi : D_R \to G$ which determines the B-reduction P_B. Recall that φ is determined up to an element of $B(R[[t]])$ and that multiplying φ for such an element does not change the reduction. By hypothesis (P_B being preserved by the connection) $\varphi'\varphi^{-1}$ belongs to the Lie algebra $\mathfrak{b} \otimes R[[t]]$, this shows by exponentiation as in the previous lemma that φ is determined by $\varphi(0)$ and that $\varphi \in \varphi(0)B(R[[t]])$. Given two different maps φ and φ' that define the same reduction on the 0 point (i.e. $\varphi(0)B(R) = \varphi'(0)B(R)$ we of course have $\varphi B(R[[t]]) = \varphi(0)B(R[[t]]) = \varphi'(0)B(R[[t]]) = \varphi' B(R[[t]])$ so they define the same reduction on D_R.

Vice versa given a fixed B-reduction of $\mathrm{Spec}\, R \times G$ determined by $\psi \in G(R)/B(R)$ we can consider the constant reduction defined by $\psi \in G(R) \subset G(R[[t]])$ the choice of the representative does not change the reduction by the above discussion. □

Thanks to the above proposition we gain the information that in the definition of a Miura Oper, given the underlying Oper (P, ∇, P_B) the set of B_--reductions preserved by ∇ are in natural correspondence with the set of B_- reductions of P_0. We are now going to make this remark more precise.

By theorem 8.5.1 the functor of Opers is representable by an affine $\mathbb{C}$-scheme $\mathrm{Op}_G(D)$, let $\mathbb{C}[\mathrm{Op}_G(D)]$ be its ring of functions and $D_{\mathrm{Op}_G(D)}$ the $\mathbb{C}[\mathrm{Op}_G(D)]$ formal disc. By the Yoneda lemma there exists a universal triple (up to isomorphism) $(P^{univ}, \nabla^{univ}, P_B^{univ})$ which is a $\mathbb{C}[\mathrm{Op}_G(D)]$-Oper, so P^{univ} is a principal trivial G-bundle on $D_{\mathrm{Op}_G(D)}$, P_B^{univ} is a B-reduction which is a trivial B-bundle and ∇^{univ} is a connection on P^{univ} which satisfies the Oper condition.

The universal property of the universal triple is the following:

Given an isomorphism class of R-Opers (P, ∇, P_B) consider the associated R point $\mathrm{Spec}\, R \to \mathrm{Op}_G(D)$ then isomorphism class of the pullback through the induced map $D_R \to D_{\mathrm{Op}_G(D)}$ of the universal triple is exactly coincides with (P, ∇, P_B).

In particular let P_0^{univ} the restriction of P^{univ} to the 0 point $\mathrm{Op}_G(D) \to D_{\mathrm{Op}_G(D)}$. This is a principal trivial G-bundle over the space of Opers. It satisfies the property that given an R-Oper (P, ∇, P_B) the pullback of P_0^{univ} is exactly P_0. In other words the fiber of the map

$$P_0^{univ}(R) \to \mathrm{Op}_G(D)(R)$$

at the Oper (P, ∇, P_B) is in natural correspondence with $P_0(R)$.

Proposition 8.6.2. *The functor*

$$R \mapsto P_0^{univ}(R) \times_{G(R)} G(R)/B_-(R)$$

is naturally isomorphic to $\mathrm{MOp}_G(D)$.

Proof. Indeed consider both functors as fibering over the functor of Opers. Combining proposition 8.6.1 and proposition 8.1.1 we see that the fiber of $\mathrm{MOp}_G(D)(R)$ at a given R-Oper (P, ∇, P_B) is canonically identified with

$$P_0(R) \times_{G(R)} G(R)/B_-(R)$$

But this is exactly the fiber of $P_0^{univ}(R) \times_{G(R)} G(R)/B_-(R)$ at the same Oper. □

Corollary 8.6.1. *The sheaf associated to* $\mathrm{MOp}_G(D)$ *is representable by*

$$P_0^{univ} \times_G G/B$$

Proof. Indeed $R \mapsto P_0(R) \times_{G(R)} G(R)/B_-(R)$ is a fat subfunctor of $P_0^{univ} \times_G G/B$. □

This gives us a nice description of the space of Miura Opers. We now turn our attention towards **generic** Miura Opers.

Definition 8.6.3. Two reductions P_B and P_{B_-} which are trivial bundles of a trivial G-bundle P are said to be in **generic** relative position if given any trivialization such that P_B is identified with $X \times B \subset X \times G$ the morphism φ_- defining the reduction has image contained in the open $BB_- \subset G$.

Recall first that the open set BB_- is isomorphic to $N \times B_-$ through the multiplication morphism, this implies for instance that for every $\mathbb{C}$-algebra R we have $BB_-(R) = B(R)B_-(R)$. This definition is independent of the choice of the trivialization and on the choice of φ_-. Indeed fix a trivialization, then the set of morphism that gives the same B_- reduction is $\varphi_- B(R[[t]])$ so the condition $\varphi_- \in BB_-(R)$ depends only on the B_--reduction and not on φ_-.

On the other hand changing the trivialization of $X \times G$ and asking that under this trivialization P_B is still identified with $X \times B \subset X \times G$ amounts to change φ_- by left multiplication by an element of $B(R[[t]])$ so also the definition does not depend on the choice of the trivialization.

Remark 8.6.1. Given two reductions in generic relative position of a trivial G-bundle P we have that $P_B \cap P_{B_-}$ is an H trivial reduction of P. through the embedding of $X \times H$ in $X \times G$ given by $n \in N(X)$ where n is the projection of $\varphi_- \in BB_-(X) = N(X) \times B_-(X)$ under the second projection.

Proposition 8.6.3. *Let* (P, P_B) *a couple of trivial* G *and* B *bundles over* X *respectively, with* P_B *a* B *reduction of* G*. Then the set of* B_- *reductions of* P *which are in generic relative position to* P_B *is canonically isomorphic to*

$$(P_B)_X(X) \times_{B(X)} BB_-(X)/B_-(X)$$

Where $(P_B)_X(X)$ *is the set of sections of the projection map* $P_B \to X$*.*

Proof. We have a canonical isomorphism $P_X(X) = (P_B)_X(X) \times_{B(X)} G(X)$, induced by the inclusion $P_B \subset P$. The set of B_- reductions is canonically isomorphic to $P_X(X)/B_-(X)$ and hence to

$$(P_B)_X(X) \times_{B(X)} G(X)/B(X)$$

It is easy to see (choosing a trivialization of P_B for instance) to show that the subset of reductions in generic relative position with P_B is exactly the subset

$$(P_B)_X(X) \times_{B(X)} BB_-(X)/B(X)$$

□

Proof. Let's show this at the level of functors. Fix a trivialization of P such that P_B is sent to $X \times B \subset X \times G$. Fix a point $p \in X(R)$ and consider the R-fibers of P_B and P_{B_-} over that point. They are respectively

$$P_{B,p} = \{(p, x) : x \in B(R)\} \qquad P_{B_-,p} = \{(p, \varphi_-(p)y) : y \in B_-(R)\}$$

Their intersection is given by the points (p, x) such that $x \in B(R) \cap \varphi_-(p)B_-(R)$, now pick the unique elements $n \in N(R)$ and $b_- \in B_-(R)$ such that $\varphi(p) = nb_-$. It is easy to see that the intersection above is just $nH(R)$. □

Definition 8.6.4. A Miura Oper $(P, \nabla, P_B, P_{B_-})$ is called **generic** if the reductions P_B and P_{B_-} are in generic relative position.

Let's make some remarks about the structure of isomorphism classes of generic Miura Opers.

Choose a trivialization of (P, P_B) such that they are identified with the trivial bundles $D_R \times B \subset D_R \times G$. The connection ∇ is by the Oper condition in the space $d + (\sum_i f_i \psi_i + \mathfrak{b} \otimes R[[t]])dt$. On the other hand since the B_- reduction is in generic position with $D_R \times B$ it is defined by a map $\varphi : D_R \to BB_-$, since changing by right multiplication with elements in $B_-(R[[t]])$ we may assume as well that $\varphi \in N(R[[t]])$. Thus we may find a trivialization (namely the one given by φ^{-1}) that preserves the B reduction and for which we may trivialize also the B_- reduction.

We just proved that any generic Miura Oper the triple $(P.P_B, P_{B_-})$ is isomorphic to the trivial triple $(D_R \times G, D_R \times B, D_R \times B_-)$. Under this trivialization the connection must be of the form

$$\nabla = d + (\sum_i f_i \psi_i + \mathbf{u}(t))dt \quad \text{with } \mathbf{u}(t) \in \mathfrak{h} \otimes R[[t]]$$

Now suppose that we have an automorphism of the triple $(D_R \times G, D_R \times B, D_R \times B_-)$. Such an isomorphism is defined by a function $\varphi \in G(R[[t]])$ and the condition that both reduction are preserved means that $\varphi \in B(R[[t]]) \cap B_-(R[[t]]) = H(R[[t]])$. So the only group acting is $H(R[[t]])$. As in the case of Opers, then the connection is uniquely expressible as

$$\nabla = d + (\sum_i f_i + \mathbf{u}(t))dt = d + (p_{-1} + \mathbf{u}(t))dt \quad \text{with } \mathbf{u}(t) \in \mathfrak{h} \otimes R[[t]] \tag{8.1}$$

This discussion proves the following proposition.

Proposition 8.6.4. *The functor of generic Miura Opers* $MOp_G(D)_{gen}$ *is isomorphic to the functor*

$$Conn(H^{p^\vee})(R) := \left\{ d + (p_{-1} + \boldsymbol{u}(t))dt \quad \textit{with } \boldsymbol{u}(t) \in \mathfrak{h} \otimes R[[t]] \right\}$$

It is therefore representable.

Proof. The natural map that on R-points sends an element $d + (p_{-1} + u(t))dt$ to the generic Miura Oper $(D_R \times G, d + (p_{-1} + u(t))dt, D_R \times B, D_R \times B_-)$ is an isomorphism by the above discussion. ☐

Using the above isomorphism we can describe a preferred system of coordinates for $MOp_G(D)_{gen}$.

Proposition 8.6.5. *Let* $\boldsymbol{b}_{i,n}$ *the function on* $MOp_G(D)_{gen}$ *defined at the level of functors*

$$\boldsymbol{b}_{i,n}(R) : MOp_G(D)_{gen}(R) \to \mathbb{A}^1(R) \qquad d + (p_{-1} + \boldsymbol{u}(t))dt \mapsto \alpha_i(\boldsymbol{u}(t))_n$$

Where by α_i *we mean the* $R[[t]]$ *linear extension of* $\alpha_i : \mathfrak{h} \to \mathbb{C}$ *while for an element* $r(t) \in R[[t]]$, $r(t)_n \in R$ *is the* n*-th coefficient, satisfying* $r(t) = \sum_{n<0} r(t)_n t^{-n-1}$*.Then the algebra of regular functions on* $MOp_G(D)_{gen}$ *is isomorphic to the free polynomial algebra generated by the* $\boldsymbol{b}_{i,n}$*.*

$$\mathbb{C}[MOp_G(D)_{gen}] = \mathbb{C}[\boldsymbol{b}_{i,n}]_{i=1,\ldots,l,n<0}$$

Proof. Since every $\mathbf{u}(t) \in \mathfrak{h} \otimes R[[t]]$ admits a unique expression

$$\mathbf{u}(t) = \sum_{i=1,\ldots,l} u_i(t)\omega_i^\vee$$

and since $\alpha_i(\mathbf{u}(t)) = u_i(t)$ it is easy to see that $MOp_G(D)_{gen}$ is isomorphic to an infinite product of copies of $\mathbb{A}^1$ and that the $\mathbf{b}_{i,n}$ introduced above are exactly the coordinates for this space. ☐

8.7 $\mathrm{Aut}\,\mathcal{O}$ and $\mathrm{Der}\,\mathcal{O}$ action on $\mathrm{MOp}_G(D)_{gen}$

As in the case of the space of Opers, a choice of preferred representatives induces an action of the group $\mathrm{Aut}\,\mathcal{O}$ on the space of Miura Opers, more precisely on the functor $\mathrm{Conn}(H^{p^\vee})$ we easily describe it as follows. Consider an element $\rho \in \mathrm{Aut}\,\mathcal{O}(R)$ and a generic R-Miura Oper $d + (p_{-1} + \mathbf{u}(t))dt$. As we saw before we have that ρ acts on such a connection by

$$\rho(t) \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big) = d + (p_{-1}\rho'(t) + \mathbf{u}(\rho(t)\rho'(t))dt$$

To compute the action on our preferred system of coordinates we have to bring this connection in the form 8.1 and this is easily achieved by applying the gauge action of $H(R[[t]])$. In particular if we consider the gauge action of $p^\vee \rho'(t)$ we get

$$p^\vee\rho'(t) \cdot \big(d + (p_{-1}\rho'(t) + \mathbf{u}(\rho(t)\rho'(t))dt\big) = d + \big(p_{-1} + \mathrm{Ad}_{p^\vee\rho'(t)}\big(\mathbf{u}(\rho(t))\rho'(t)\big) - p^\vee \frac{\rho''(t)}{\rho'(t)}\big)dt$$

And since H is commutative $\mathrm{Ad}_{p^\vee\rho'(t)}$ acts trivially on $\mathfrak{h} \otimes R[[t]]$. We obtain therefore the following proposition.

Proposition 8.7.1. *The action of* $\mathrm{Aut}\,\mathcal{O}$ *on* $\mathrm{MOp}_G(D)_{gen}$ *or equivalently on* $\mathrm{Conn}(H^{p^\vee})$ *is given by the following formula:*

$$\rho(t) \cdot \big(d + (p_{-1} + \boldsymbol{u}(t))dt\big) = d + \big(p_{-1} + \boldsymbol{u}(\rho(t))\rho'(t) - p^\vee \frac{\rho''(t)}{\rho'(t)}\big)dt$$

8.7.1 Action of $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$ on the ring of functions

Now that we described the action of $\mathrm{Aut}\,\mathcal{O}$ on $\mathrm{MOp}_G(D)_{gen}$ we can also describe the action of $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$ on the ring of functions $\mathbb{C}[\mathrm{MOp}_G(D)_{gen}]$. We are particularly interested in the action of the operators $L_n = -t^{n+1}\partial_t = t + \epsilon t^{n+1}$, $n \geq -1$ which are topological generators of $\mathrm{Der}\,\mathcal{O}(\mathbb{C})$.

Proposition 8.7.2. *The operators* $L_n = -t^{n+1}\partial_t$ *act as on* $\mathbb{C}[\mathrm{MOp}_G(D)_{gen}]$ *as derivations. On the generators* $\boldsymbol{b}_{i,m}$ *they act according to the following formulas:*

$$\begin{aligned} &L_n \cdot \boldsymbol{b}_{i,m} = -m\boldsymbol{b}_{i,n+m} \;\; \text{if} \;\; -1 \leq n < -m \\ &L_n \cdot \boldsymbol{b}_{i,-n} = -n(n+1) \;\; \text{if} \;\; n > 0 \\ &L_n \cdot \boldsymbol{b}_{i,m} = 0 \;\; \text{if} \;\; n > -m \end{aligned}$$

In particular L_0 *acts on* $\mathbb{C}[\mathrm{MOp}_G(D)_{gen}]$ *semisimply with integer eigenvalues and its character under this action is given by*

$$\mathrm{ch}(\mathbb{C}[\mathrm{MOp}_G(D)_{gen}]) = \prod_{i=1,\ldots,l} \prod_{n<0} \frac{1}{1-q^n} \tag{8.2}$$

Proof. To compute the action of L_n we have to apply the action of $t - \epsilon t^{n+1} \in \mathrm{Aut}\,\mathcal{O}(\mathbb{C}[\epsilon])$ on the function $\mathbf{b}_{i,m}$ (this amounts to precomposing $\mathbf{b}_{i,m}$ with $(t-\epsilon t^{n+1})^{-1} = t + \epsilon t^{n+1}$) and then take the derivative with respect to ϵ. Thus we consider

$$\mathbf{b}_{i,m} : \mathrm{Conn}(H^{p^\vee})(R) \to R \qquad d + (p_{-1} + \mathbf{u}(t))dt = d + (p_{-1} + \sum_i \omega_i^\vee \sum_{n<0} u_{i,n} t^{-n-1})dt \mapsto u_{i,m}$$

And we precompose with the action of $(t-\epsilon t^{n+1})^{-1}$

$$
\begin{aligned}
(t+\epsilon t^{n+1})\cdot\big(d+\big(p_{-1}+\mathbf{u}(t)\big)dt\big) &= \\
= d+\left(p_{-1}+\mathbf{u}(t+\epsilon t^{n+1})(1+(n+1)\epsilon t^{n})-p^{\vee}\frac{(n+1)n\epsilon t^{n-1}}{1+(n+1)\epsilon t^{n}}\right)dt \\
= d+(p_{-1}+\mathbf{u}(t))dt+\epsilon\big(t^{n+1}\mathbf{u}'(t)+(n+1)\mathbf{u}(t)t^{n}-p^{\vee}(n+1)nt^{n-1}\big)dt
\end{aligned}
$$

Now we write $\mathbf{u}(t)=\sum_i \omega_i^{\vee}\sum_{n<0}u_{i,n}t^{-n-1}$ so that

$$
\begin{aligned}
t^{n+1}\mathbf{u}'(t)+(n+1)\mathbf{u}(t)t^{n}-p^{\vee}(n+1)nt^{n-1} &= \\
\sum_i \omega_i^{\vee}\sum_{m<0}\big((-m-n-1)u_{i,m+n}+(n+1)u_{i,m+n}-\delta_{m,-n}n(n+1)\big)t^{-m-1} &= \\
\sum_i \omega_i^{\vee}\sum_{m<0}\big((-m)u_{i,m+n}-\delta_{m,-n}n(n+1)\big)t^{-m-1}
\end{aligned}
$$

Where for $m\geq 0$ we set $u_{i,m}=0$.

We therefore have

$$
\mathbf{b}_{i,m}\Big((t+\epsilon t^{n+1})\cdot\big(d+\big(p_{-1}+\mathbf{u}(t)\big)dt\big)\Big)=u_{i,m}+\epsilon\big(-mu_{i,m+n}-\delta_{m,-n}n(n+1)\big)
$$

Taking the derivative gives now the desired formulas. □

Theorem 8.7.1. *The isomorphism of commutative algebras*

$$
\pi_0({}^L\mathfrak{g})\to\mathbb{C}[MOp_G(D)_{gen}]\qquad b_{i,n}\mapsto -\boldsymbol{b}_{i,n}
$$

is $\mathrm{Der}\,\mathcal{O}$ *equivariant.*

Proof. This is obvious thanks to formulas of proposition 5.6.4 after noticing that the description of $MOp_G(D)_{gen}$ we gave do not change if we replace G with ${}^LG$. □

8.8 Another description of $MOp_G(D)_{gen}$

We consider now a different, more geometric, description of generic Miura Opers. This description will allow us to show that they form a N-torsor over the space of Opers.

Consider the fiber of the forgetful morphism $MOp_G(D)_{gen}(R)\to Op_G(D)(R)$ over a given R-Oper (P,∇,P_B). Now the choice of a B_- reduction preserved by ∇ is equivalent, by proposition 8.6.1, to the choice of a B_- reduction of P_0. The additional condition to be in generic relative position to P_B ma be easily expressed in terms of the relative position of $P_{B,0}$ and $P_{B_-,0}$.

Remark 8.8.1. Consider two reductions of the trivial bundle $D_R\times G$ for the subgroups B and B_-. Then they are in generic relative position if and only if their restrictions to the 0 point are in generic relative position. Indeed we may suppose that $P_B=D_R\times B\subset D_R\times G$, then the reduction P_{B_-} is in generic relative position with $D_R\times B$ if and only if any defining map $\varphi: D_R\to G$ (for P_{B_-}) takes image in the open $BB_-\subset G$. As we saw in the preliminaries this is equivalent to ask that $\varphi(0)$ takes image in BB_-, but again, this is equivalent to ask that $P_{B,0}$ and $P_{B_-,0}$ are in generic relative position.

Now combining the previous remark with proposition 8.6.3 we obtain that the B_- reductions of a given R-Oper (P, ∇, P_B) are in natural bijection with the set

$$(P_{B,0})_{\mathrm{Spec}\,R}(R) \times_{B(R)} BB_-(R)/B_-(R)$$

Call $\mathcal{U}$ the open B-orbit $U := BB_-/B_- \subset G/B_-$. It is an affine scheme isomorphic to N. Combining the above remarks and a proof analogous to the one of corollary 8.6.1 we obtain the following proposition.

Proposition 8.8.1. *The space of generic Miura Opers is isomorphic to:*

$$MOp_G(D)_{gen} = P_{B,0}^{univ} \times_B U$$

In particular choosing a trivialization of P_B^{univ} which induces a trivialization of $P_{B,0}^{univ}$ we obtain that

$$MOp_G(D)_{gen} \simeq Op_G(D) \times U \simeq Op_G(D) \times N$$

Let's describe more precisely this isomorphism at the level of functors.

Proposition 8.8.2. *The functor of* $\mathbb{C}$ *algebras*

$$R \mapsto \left\{(D_R \times G, d+(p_{-1}+v(t))dt, D_R \times B, P_{B_-}) : v(t) \in V \otimes R[[t]] \text{ and the quadruple is a generic Miura Oper } \right\}$$

Is isomorphic to the space of generic Miura Opers and to $Op_G(D) \times N$.

Proof. Since the principal bundles $D_R \times G$ and $D_R \times B$ are already trivialized and P_{B_-} is a trivial principal B_- bundle in generic position with $D_R \times B$. The set of such reductions is in correspondence with morphisms $\varphi \in BB_-(R[[t]])/B(R[[t]]) = N(R[[t]])$. The condition of being a Miura Oper in this case is equivalent to ask that P_{B_-} is preserved by the connection. By proposition 8.6.1 the set of morphisms $\varphi \in N(R[[t]])$ that preserve the connection is in correspondence with $N(R)$. This shows that the functor defined above is isomorphic to $Op_G(D) \times N$. On the other hand we have a natural map from our functor to the functor of generic Miura Opers. Trivializing P_{B_-} under the isomorphism induced by its defining morphism $\varphi \in N(R[[t]])$ brings the quadruple into the form of proposition 8.6.4 since the gauge action of $N(R[[t]])$ do not change the factor p_{-1}. And therefore the quadruples $(D_R \times G, d + (p_{-1} + v(t))dt, D_R \times B, P_{B_-})$ are in correspondence with isomorphism classes of generic Miura Opers. This shows that the functor above is isomorphic also to $MOp_G(D)_{gen}$. □

After choosing such a trivialization we therefore obtain a left N-action on $MOp_G(D)_{gen}$. Since the latter space is a product we obtain that $Op_G(D) \simeq MOp_G(D)_{gen}/N$.

The goal of the following section is to describe this action.

8.9 Action of N on the space of generic Miura Opers

We now investigate an action of the group N on the space of generic Miura Opers. The description given in proposition 8.8.1 will allow us the describe $Op_G(D)$ with the quotient of $MOp_G(D)_{gen}$ by the action of N cited above. Finally we are going to be able to identify the algebra of functions on $Op_G(D)$ as the subspace of $\mathfrak{n}$-invariants of the algebra of functions on $MOp_G(D)_{gen}$.

Consider a trivialized generic R-Miura Oper $(D_R \times G, d + (p_{-1} + u(t))dt, D_R \times B, D_R \times B_-)$. Let's describe the left action of N under the isomorphism of proposition 8.8.2. Intrinsically, since N is acting on the left and the factor 'N' in $MOp_G(D) \times N$ stands for the map defining the second reduction $P_{B_-,0}$ this action amounts exactly to translating with N the trivialization of $P_{B_-,0}$. Indeed we can change the B_- reduction $\operatorname{Spec} R \times B_-$ by multiplying on the left by an element $x \in N(R)$. The B_- reduction we obtain in this way is still in generic relative position with $\operatorname{Spec} R \times B$ and therefore by what we have done so far it corresponds to a unique B_- reduction $P_{B_-,x}$ of $D_R \times G$ that is still in generic relative position with $D_R \times B$. All B_- reductions preserved by the connection and in generic relative position with $D_R \times B$ are obtained in this way. This generates the $N(R)$ action over the space of R-Miura Opers we wanted to describe.

We are now going to describe this action on the functor $\mathcal{C}onn(H^{p^\vee})$. We are given a trivialized R-Miura Oper

$$(D_R \times G, d + (p_{-1} + \mathbf{u}(t))dt, D_R \times B, D_R \times B_-)$$

The left action of $x \in N(R)$ brings this Miura Oper to the Miura Oper

$$(D_R \times G, d + (p_{-1} + \mathbf{u}(t))dt, D_R \times B, P_{B_-,x})$$

where $P_{B_-,x}$ is the B_- reduction of $D_R \times G$ induced by the unique element $x(t) \in N(R[[t]])$ such that $P_{B_-,x}$ is preserved by ∇, or in other words, such that:

$$x(t) \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big) \in d + \big(p_{-1} + \mathfrak{h} \otimes R[[t]]\big)dt$$

Now, in order to re-trivialize $(D_R \times G, \nabla, D_R \times B, P_{B_-,x})$ into the form $(D_R \times G, \nabla', D_R \times B, D_R \times B_-)$ we have to consider the isomorphism induced by $x(t)^{-1}$. The connection is brought to the form

$$x(t) \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big)$$

We proved the following: the action of $x \in N(R)$ on $\mathbf{u}(t) \in \mathfrak{h} \otimes R[[t]]$ is given by the only $(x \cdot \mathbf{u})(t) \in \mathfrak{h} \otimes R[[t]]$ such that

$$d + (p_{-1} + (x \cdot \mathbf{u})(t))dt = x(t) \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big)$$

where $x(t) \in N(R[[t]])$ is the unique element such that $x(0) = x$ and such that the right hand side belongs to $d + (p_{-1} + \mathfrak{h} \otimes R[[t]])dt$.

8.9.1 Infinitesimal action of $\mathfrak{n}$

We now study the infinitesimal action of $\mathfrak{n}$ on this space. We will focus on the generators of $\mathfrak{n}$ given by the fixed root space decomposition $e_i \in \mathfrak{g}_{\alpha_i}$. Consider an $e_i \in \mathfrak{n}(\mathbb{C}) \subset N(\mathbb{C}[\epsilon])$ and an element $d + (p_{-1} + \mathbf{u}(t))dt \in \mathcal{C}onn(H^{p^\vee})(R)$. To calculate the action of e_i we think as the latter elements as belonging to $\mathcal{C}onn(H^{p^\vee})(R)$.

Let $x_i(t) \in R[[t]]$ an element such that $x_i(0) = 1$ and such that

$$[x_i(t)e_i, p_{-1} + \mathbf{u}(t)] - \partial_t x_i(t) \in \mathfrak{h} \otimes R[[t]]$$

If we write $\mathbf{u}(t) = \sum_i u_i(t)\omega_i^\vee$ where $u_i(t) \in R[[t]]$ and $\omega_i^\vee$ is defined by $\alpha_j(\omega_i^\vee) = \delta_{ij}$. The equation reads as follows:

$$x_i(t)h_i - x_i(t)u_i(t)e_i - \partial_t x_i(t)e_i \in \mathfrak{h} \otimes R[[t]]$$

So $x_i(t)$ must satisfy $\partial_t x_i(t) = -x_i(t)u_i(t)$ and therefore is uniquely determined by its value on 0. In particular the following expression holds for $x_i(t)$.

$$x_i(t) = \sum_{n<0} x_{i,n} t^{-n-1} = \exp\Big(\sum_{m<0} \frac{u_{i,m}}{m} t^{-m}\Big) \tag{8.3}$$

Proposition 8.9.1. *The unique element* $e_i(t) \in N(R[[t]][\epsilon_0])$ *such that*

$$e_i(t) \cdot \big(d + (p_{-1} + \boldsymbol{u}(t))dt\big) \in d + (p_{-1} + \mathfrak{h} \otimes R[[t]])dt$$

Is $x(t)e_i$.

Proof. Indeed the action of $x_i(t)e_i \in N(R[[t]][\epsilon])$ is given by

$$\begin{aligned} x_i(t)e_i \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big) = d + \big(\mathrm{Ad}_{x_i(t)e_i}(p_{-1} + \mathbf{u}(t))\big)dt - \epsilon\partial_t x_i(t)e_i dt = \\ d + (p_{-1} + \mathbf{u}(t))dt + \epsilon\big([x_i(t)e_i, p_{-1} + \mathbf{u}(t)] - \partial_t x_i(t)e_i\big)dt \end{aligned}$$

And by construction of $x_i(t)$ it belongs to the space $d + (p_{-1} + \mathfrak{h} \otimes R[[t]][\epsilon])dt$ as requested. □

We now investigate the action of $\mathfrak{n}$ on the ring of functions on $\mathrm{MOp}_G(D)_{gen}$.

We look at the system of coordinates for $\mathrm{MOp}_G(D)_{gen}$ introduced in proposition 8.6.5. We are now ready to see how $e_i \in \mathfrak{n}$ acts on such functions.

Proposition 8.9.2. *The action of* e_i *as a vector field over* $\mathrm{MOp}_G(D)_{gen}$ *is given by the following formula:*

$$e_i = -\sum_{j=1,\dots l} a_{ij} \sum_{n<0} x_{i,n} \frac{\partial}{\partial \boldsymbol{b}_{j,n}} \tag{8.4}$$

where $x_{i,n}$ *is given by formula 8.3 making the substitution* $u_{j,n} \mapsto \boldsymbol{b}_{j,n}$ *and it is a polynomial in the coordinates* $b_{j,n}$.

Proof. We already now that the action of $\mathfrak{n}$ is given by vector fields. Since a $\mathbb{C}[\mathbf{b}_{i,n}]$ basis for the vector fields is given by the derivations $\frac{\partial}{\partial \mathbf{b}_{j,n}}$ we obtain that

$$e_i = \sum_{j=1,\dots,l,n<0} (e_i \cdot \mathbf{b}_{j,n}) \frac{\partial}{\partial \mathbf{b}_{j,n}}$$

and we can restrict ourselves to compute the action of e_i on the coordinate functions $\mathbf{b}_{j,n}$. We will compute the action of e_i on the R points of the scheme, the following calculations provide therefore the action of $-e_i$ on the ring of functions.

By definition the action of e_i on $\mathrm{MOp}_G(D)(R)$ is given by the formula

$$\begin{aligned} e_i \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big) &= x(t)e_i \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big) \\ = d + (p_{-1} + \mathbf{u}(t))dt + \epsilon\big([x(t)e_i, p_{-1} + \mathbf{u}(t)] - \partial_t x(t)e_i\big)dt \\ = d + (p_{-1} + \mathbf{u}(t))dt + \epsilon x_i(t)h_i \end{aligned}$$

Where the last equality follows from the fact that $x_i(t)$ is chosen in order to make $[x_i(t)e_i, p_{-1} + \mathbf{u}(t)] - \partial_t x_i(t)e_i = [x_i(t)e_i, p_{-1}] = x_i(t)h_i$. Now by definition the action of e_i on $\mathbf{b}_{j,n}$ is computed by evaluating this point with $\mathbf{b}_{j,n}$ to obtain a $R[\epsilon]$ point of $\mathbb{A}^1$, taking the derivative with respect to ϵ and finally evaluating at $\epsilon = 0$.

$$\mathbf{b}_{j,n}\Big(e_i \cdot \big(d + (p_{-1} + \mathbf{u}(t))dt\big)\Big) = \mathbf{b}_{j,n}\big(d + (p_{-1} + \mathbf{u}(t))dt + \epsilon x_i(t)h_i dt\big) = u_{j,n} + \epsilon\alpha_j(h_i)x_{i,n}$$

Then taking the derivative with respect to ϵ gives us exactly

$$e_i \cdot \mathbf{b}_{j,n} = -\alpha_j(h_i)x_{i,n} = a_{ij}x_{i,n}$$

as required. □

This description allows us to describe the algebra of functions on $\mathrm{Op}_G(D)$ as the intersection of the operators e_i.

Theorem 8.9.1. *The algebra of functions on the space of Opers on the disc* $\mathrm{Op}_G(D)$ *equals the intersection of the kernels of the operators* e_i

$$\mathbb{C}[\mathrm{Op}_G(D)] = \bigcap_{i=1,\dots,l} \ker(e_i)$$

Proof. Since $\mathrm{MOp}_G(D)_{gen}$ is isomorphic to the trivial N-principal bundle over $\mathrm{Op}_G(D)$. We have

$$\mathbb{C}[\mathrm{Op}_G(D)] = \mathbb{C}[\mathrm{MOp}_G(D)_{gen}]^N = \mathbb{C}[\mathrm{MOp}_G(D)_{gen}]^{\mathfrak{n}} = \bigcap_{i=1,\dots,l} \ker(e_i)$$

where the last equality follows from the fact that the e_i generates $\mathfrak{n}$ while the other equalities follow from the the fact $\mathrm{MOp}_G(D)_{gen} \simeq \mathrm{Op}_G(D) \times N$. Indeed N acts only on the second factor, and hence also on the second factor of the decomposition

$$\mathbb{C}[\mathrm{MOp}_G(D)_{gen}] = \mathbb{C}[\mathrm{Op}_G(D)] \otimes \mathbb{C}[N]$$

The same is true for the action of $\mathfrak{n}$. Since the ring of invariant function for both actions is the ring of constant functions the theorem is proved. □

8.10 Computation of the character

Our last goal is to compute the character under the action of the operator $L_0 = t\partial_t$ of $\mathbb{C}[\mathrm{Op}_G(D)]$. Consider the isomorphism $\mathrm{MOp}_G(D)_{gen} \simeq \mathrm{Op}_G(D) \times N$ we introduced in proposition 8.8.2. Recall that under this isomorphism N acts by left multiplication on the second factor. Always under this isomorphism we have

$$\mathbb{C}[\mathrm{MOp}_G(D)] = \mathbb{C}[\mathrm{Op}_G(D)] \otimes \mathbb{C}[N]$$

and the action of $N(\mathbb{C})$ on this ring of functions is only on the second factor. On the other hand $\mathfrak{n}(\mathbb{C})$ acts by derivations and acts like 0 on the first factor $\mathbb{C}[\mathrm{Op}_G(D)] \otimes 1$. The grading operator L_0 acts by derivations as well.

It is reasonable to think that taken coordinates for N y_α such that $e_\alpha \cdot y_\alpha = 1$ they must have degree $\mathrm{ht}(\alpha)$. We show this in detail:

Proposition 8.10.1. *The Operator* $L_0 = -t\partial_t$ *preserves the subalgebra* $\mathbb{C}[N]$. *And acts like the derivation induced by the adjoint action of* $p^\vee(1+\epsilon) \in H(\mathbb{C}[\epsilon])$.

Proof. In order to do prove this proposition we keep considering the isomorphism of proposition 8.8.2.

Under this description consider the generic R-Miura Oper $(D_R \times G, d+(p_{-1}+\nu(t))dt, D_R \times B, x)$ where $x \in N(R)$ determines the unique B_- reduction preserved by the connection. We view this as an $R[\epsilon]$ generic Miura Oper, then applying the left action of $t + \epsilon t$ amounts to changing the coordinate in the connection but does not effect the B_- reduction. Thus:

$$(t+\epsilon t)\cdot(D_R \times G, d+(p_{-1}+\nu(t))dt, D_R \times B, x) = (D_R \times G, d+\big(p_{-1}(1+\epsilon)+\nu(t+\epsilon t)(1+\epsilon)\big)dt, D_R \times B, x)$$

To bring it again in the our canonical form we must change trivialization. It suffices to change trivialization with the element $p^\vee(1+\epsilon) \in H(\mathbb{C}[\epsilon]) \subset H(R[\epsilon])$. This brings the connection in the form

$$p^\vee(1+\epsilon)\cdot\left(d + \big(p_{-1}(1+\epsilon) + \nu(t+\epsilon t)(1+\epsilon)\big)dt\right) = d + \left(p_{-1} + Ad_{p^\vee(1+\epsilon)}\big(\nu(t)(1+\epsilon)\big)\right)dt$$

This equality follows from the fact that $(t+\epsilon t)'' = 0$. Notice that since $\nu(t) \in V \otimes R[[t]]$ we have that $Ad_{p^\vee(1+\epsilon)}\big(\nu(t)(1+\epsilon)\big) \in V \otimes R[[t]]$ so the connection is brought in the canonical form. This change of trivialization brings the B_- reduction in the B_- reduction associated to the point $(p^\vee(1+\epsilon))x \in BB_-(R[\epsilon)/B_-(R[\epsilon])$ its representative in $N(R[\epsilon])$ is exactly

$$Ad_{p^\vee(1+\epsilon)}(x) \in N(R[\epsilon])$$

This proves the second part of the proposition, but it easily implies the first one.

Indeed since we showed that $(t+\epsilon t)$ acts separately on the two factors of the product it is evident that taking the derivative of a function constant on the first factor yields a function of the same form. □

Corollary 8.10.1. *The character of the subalgebra* $\mathbb{C}[N]$ *under the operator* $L_0 = -t\partial_t$ *is given by the following formula.*

$$ch(\mathbb{C}[N]) = \prod_{\alpha\in\Phi^+} \frac{1}{1-q^{p^\vee(\alpha)}} = \prod_{i=1}^{l}\prod_{n_i=1}^{d_i} \frac{1}{1-q^{n_i}}$$

Proof. Consider the H-equivariant isomorphism $exp : \mathfrak{n} \to N$, where H acts on both spaces through the adjoint action. We can restrict ourselves to computing the character of L_0, which acts like an element of H on the first space.

Consider a coordinate function on $\mathfrak{n}$, $y_\beta : \mathfrak{n} \to \mathbb{A}^1$. Then for any R-point of $\mathfrak{n}$, $\sum_\alpha r_\alpha e_\alpha$ we have

$$y_\beta\big(p^\vee(1+\epsilon)\cdot(\sum_\alpha r_\alpha e_\alpha)\big) = y_\beta\big(\sum_\alpha r_\alpha e_\alpha + \epsilon(\sum_\alpha ht(\alpha) r_\alpha e_\alpha)\big) = r_\beta + \epsilon ht(\beta) r_\beta$$

And we immediately get $L_0 \cdot y_\beta = ht(\beta) y_\beta$. The first part of the equation immediately follows while the second equality follows from the discussion in the introduction about the exponents of $\mathfrak{g}$. □

Corollary 8.10.2. *The character for* $\mathbb{C}[\mathrm{Op}_G(D)]$ *under the action of* L_0 *is given by the following formula.*

$$\mathrm{ch}(\mathbb{C}[\mathrm{Op}_G(D)]) = \prod_{i=1}^{l} \prod_{n \geq d_i+1} \frac{1}{1-q^n} \tag{8.5}$$

Proof. This is a straightforward computation. We have $\mathrm{ch}(\mathbb{C}[\mathrm{MOp}_G(D)_{gen}] = \prod_{i=1}^{l} \prod_{n>0} \frac{1}{1-q^n}$ and

$$\mathrm{ch}(\mathbb{C}[\mathrm{MOp}_G(D)_{gen}]) = \mathrm{ch}(\mathbb{C}[\mathrm{Op}_G(D)])\mathrm{ch}(\mathbb{C}[N])$$

And then we use the formula of corollary 8.10.1 □

8.11 Identification with the algebra of function on Opers

We are now ready to prove theorem 4.6.1.

Consider the isomorphism of theorem 8.7.1.

$$\pi_0({}^L\mathfrak{g}) \to \mathbb{C}[\mathrm{MOp}_G(D)_{gen}]$$

Recall that we already proved in Proposition 6.2.1 that the center of the vertex algebra injectively maps in π_0. Moreover we showed in Chapter 'Screening Operators' that

$$\zeta({}^L\mathfrak{g}) \subset \bigcap_i \ker(\overline{V}_i)$$

lies in the intersection of the kernels of the screening operators. But under the isomorphism $\pi_0({}^L\mathfrak{g}) = \mathbb{C}[\mathrm{MOp}_G(D)]$, by comparing formulas in Proposition 7.5.3 and 8.4. We get that $\overline{V}_i = -e_i$, thus

$$\zeta({}^L\mathfrak{g}) \subset \bigcap_i \ker(\overline{V}_i) = \bigcap_i \ker(e_i) = \mathbb{C}[\mathrm{Op}_G(D)]$$

But we also computed the characters of both spaces under the action of L_0, and they are indeed equal.

Theorem 8.11.1. *The center of the vertex algebra* $\zeta(\mathfrak{g})$ *is isomorphic in a* $\mathrm{Der}\,\mathcal{O}$ *equivariant way to the algebra of regular functions on the space of* ${}^L G$ *Opers on the disc:* $\mathrm{Op}_{{}^L G}(D)$.

Proof. The embeddings presented above

$$\zeta({}^L\mathfrak{g}) \subset \mathbb{C}[\mathrm{MOp}_G(D)_{gen}] \qquad \mathbb{C}[\mathrm{Op}_G(D)] \subset \mathbb{C}[\mathrm{MOp}_G(D)_{gen}]$$

are $\mathrm{Der}\,\mathcal{O}$ equivariant by construction. Therefore the embedding

$$\zeta({}^L\mathfrak{g}) \subset \mathbb{C}[\mathrm{Op}_G(D)]$$

is $\mathrm{Der}\,\mathcal{O}$ equivariant as well. By comparison of formulas 6.6.2 and 8.5. We get that the characters of these spaces under the action of L_0 are indeed equal. Since the morphism is $\mathrm{Der}\,\mathcal{O}$ equivariant we obtain that

$$\zeta({}^L\mathfrak{g}) = \mathbb{C}[\mathrm{Op}_G(D)]$$

Making the substitutions ${}^L\mathfrak{g} \mapsto \mathfrak{g}$ and $G \mapsto^L G$ we conclude the proof. □

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